COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 79, nº 1 (1991), p. 63-97.

<http://www.numdam.org/item?id=CM_1991__79_1_63_0>

Compositio Mathematica **79**: 63–97, 1991.

The Milnor fiber and the zeta function of the singularities of type $f = P(h, g)$

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Received 24 October 1989; accepted 20 July 1990

1. Introduction

An important invariant of a germ of analytic function $f:(\mathbf{C}^{n+1},0)\to(\mathbf{C},0)$ is its Milnor fiber F. Milnor [21] proved that if f has an isolated singularity at the origin then the homotopy type of the fiber is a bouquet of μ spheres of dimension n, where μ is the Milnor number of f. For isolated line singularities (not of type A_∞) Siersma [29] proved that the Milnor fiber is homotopy equivalent to a bouquet of $(\#A_1+2\#D_\infty-1)$-spheres of dimension n, where $\#A_1$ and $\#D_\infty$ are the number of A_1 and, respectively D_∞ points in a generic approximation of f. More generally, if the critical set Sing $V(f)$ of f is an 1-dimensional complete intersection (ICIS) and the transversal singularity of f in the points of Sing $V(f)-\{0\}$ is of type A_1, then in the general ($\#D_\infty>0$) case F is homotopic to $\bigvee S^n$ and in the special case ($\#D_\infty=0$) F is homotopic to $S^{n-1}\vee(\bigvee S^n)$ [31]. In the case of line singularities, but with transversal type $S=A_1, A_2, A_3, D_4, E_6, E_7, E_8$ de Jong [37] proved that in the general case the Milnor fiber is homotopic to $(\bigvee S^{n-1})\vee(\bigvee S^n)$. Other important cases with 1-dimensional singular locus were studied by Iomdin [14], Pellikaan [23] and Siersma [32].

Perhaps the most efficient method in the study of the Milnor fiber of non-isolated singularities is the polar slice technique which materialized in two beautiful formulae for the Euler-characteristic of the fibers: the Lê attaching formula [16] and the Iomdin–Lê formula [14, 18]. Moreover, Lê [17] used the polar slice construction to show that the geometric monodromy of f preserves the polar filtration and can be constructed as a carrousel.

In the present paper our main object of study is an analytic germ with the following property: f can be written as $f=P(h,g)$, where the pair $(h,g):(\mathbf{C}^{n+1},0)\to(\mathbf{C}^2,0)$ defines an ICIS and P is a germ of two variables.

Note that if P has a (proper) isolated singularity at the origin then $\dim \operatorname{Sing} V(f)=n-1$ and if P is not reduced, then $\dim \operatorname{Sing} V(f)=n$. It is easy to see that in the latter case even the connectivity of the fiber may fail. But even if the fiber is connected we can not expect its simple-connectivity because of a theorem due to Kato and Matsumoto [15] giving, in general, the best bound for the connectivity of the fiber (in terms of the dimension of the singular set). But, somewhat surprisingly, by our results we do have a "depth connectivity": $\pi_k(F) = 0$ for $2 \leqslant k \leqslant n-2$.

This is perhaps the first example of a class of germs with non-separable variables for which we have important informations on the other side of the "Kato–Matsumoto bound".

More precisely, we prove the following

THEOREM A.

(a) *The Milnor fiber F has the homotopy type of a space obtained from the total space of a fiber bundle (with base space the fiber of P and with fiber the general fiber of the ICIS) by attaching $(\gamma_f(c), f)_0$ cells of dimension n.*
(b) *If $P^{-1}(0) \cap D = \{0\}$ (D denotes the reduced discriminant locus of the ICIS) then F has the homotopy type of π_0 (Milnor fiber of P) copies of a bouquet of spheres $(\bigvee S^1) \vee (\bigvee S^n)$. (See* 2.0.1 *and* 2.0.2 *for the definition of $\gamma_f(c)$).*

The number of (1-dimensional) circles and n-dimensional spheres in a connected component is determined in terms of the Milnor number of the ICIS, the topological invariants of P and an intersection number.

Theorem A contains as particular cases the Teissier [36] restriction formula, the Lê attaching formula (in the case of dim Sing $V(f) = 1$) and the Iomdin–Lê formula. This is not surprising, since in this case the polar curve technique is replaced by a method based on the properties of the discriminant space of an ICIS. During the proof, when comparing the Milnor fiber of f with its part contained in a "good representative" of the ICIS we are using a technical lemma of Iomdin [13] (Section (3.1)).

The most important topological invariant of *isolated singularities* is the Seifert form. In fact, for $n \geqslant 3$ it is an absolute invariant in virtue of the classification theorem of simple spinnable structures on S^{2n+1}, proved independently by Kato and Durfee. Unfortunately, in general, it is very hard to calculate it. For plane curve singularities it was calculated by A'Campo [1] and Gusein-Zade [11, 12]. But for $n \geqslant 2$ even in the simpler (but very important) case of quasihomogeneous isolated singularities we do not know this invariant. Moreover, it appears that even the computation of weaker invariants, such as the intersection form and the monodromy action is still a very hard problem. Concerning the zeta function there are some general results: A'Campo [2] determined it in terms of invariants of the resolution, Varchenko [38] in terms of the Newton diagram, while Milnor and Orlik [22] calculated it for quasihomogeneous isolated singularities.

In Section (2.2) we calculate (Theorem B) the zeta function of $f = P(h, g)$ in terms of the zeta function of P (which is well-known), the monodromy representation and the singular monodromies of the ICIS (the latter may be calculated using the vertical monodromies of the ICIS). In the case when the monodromy representation has an abelian image this formula becomes simpler and is related to the Alexander polynomial of the link $S^3_\eta \cap (D \cup P^{-1}(0)) \subset S^3_\eta$ (Theorem C).

In the class of all singularities, the isolated singularities play a distinguished role. Similarly, in our class of germs, there is a distinguished class: those germs having the property $D \cap P^{-1}(0) = \{0\}$. Any "bad" germ $f_\infty = P_\infty(h, g)$ (i.e. $D \cap P^{-1}_\infty(0) \neq \{0\}$) can be approximated by a series of distinguished singularities $f_k = P_k(h, g)$. This is similar to the case of Arnold's series, which approximate some non-isolated singularities, and with the case of Iomdin's series or the topological series of plane curve singularities introduced by Schrauwen [26].

The best approximation holds for "topologically trivial series", when the topological type of P_k and P_∞ agree, but their mutual position with respect to D varies. In this case we calculate the zeta function (in particular the Euler-characteristic of the Milnor fiber) of f_k in terms of the zeta function of f_∞ and the vertical and horizontal monodromies of the ICIS (Theorem D). In particular for the Iomdin's series we recover Siersma's result [33], which was, in fact, our starting point in the study of series of singularities.

The polar filtration method used by Siersma is replaced here by the technique of fibered multilinks (the decomposition of the complement space along the separating tori, determined by splice decomposition).

In Section (2.4) we prove that the complement X of a projective hypersurface given by a homogeneous polynomial $f = P(h, g)$ is $(n-1)$-equivalent to the Eilenberg MacLane space $K(\pi_1(X), 1)$. If f is "distinguished" then we have a n-equivalence, and using this correspondence we calculate the homology groups of X in terms of $\pi_1(X)$ which is also computed. This result may be regarded as a generalization of a theorem of Kato and Matsumoto [15, Proposition 4.2]. In particular this procedure can be used for a large class of singular plane curves in order to compute the fundamental group of the complement space.

In (2.5) we consider the finite I-codimension germs in the sense of Pellikaan [23, 24], where $I = (h^{k-1})$ $(k \geq 2)$, and h is a germ with an isolated singularity at the origin. By our Criterion the germs with finite I-codimension are of the form $f = h^k g$, with (h, g) ICIS and g having an isolated singularity. Our general results may be applied to this special case. For example the Milnor fiber is a bouquet of spheres $S^1 \vee (\bigvee_{\mu_n} S^n)$ where $\mu_n = k(\mu(h, g) + \mu(h)) + \mu(h, g) + \mu(g)$.

For the theory of complete intersections we refer the reader to the book of Looijenga [20], for singularity theory to the monograph Arnold–Gusein-Zade–Varchenko [3], for link theory to [9], for Section 2.5 to the thesis of Pellikaan [23] and for the proof of Theorem C to the paper by Fox [10] about the free differential calculus.

2. The main results

2.0. Preliminaries and notations

2.0.1. Throughout this paper we shall denote by $\mathcal{O}$ the local ring of germs of analytic functions $f:(\mathbf{C}^{n+1},0)\to(\mathbf{C},0)$ and by m its maximal ideal. $V(\alpha)$ denotes the germ of the zero-set of an ideal $\alpha\subset\mathcal{O}$. If α is an ideal in $\mathcal{O}$ such that $V(\alpha)$ is a curve and $f\in\mathcal{O}$ such that $V(\alpha)\cap V(f)=\{0\}$ then we use the (algebraic) intersection number $(\alpha, f)_0$ given by $\mathbf{dim}(\text{coker})-\mathbf{dim}(\text{ker})$ of the map $\mathcal{O}/\alpha \xrightarrow{\cdot f} \mathcal{O}/\alpha$ (multiplication by f). The following lemma is well-known:

LEMMA.

(a) *If* $k > \mathbf{dim}_{\mathbf{C}}\,\mathcal{O}/\alpha+(f)$, *then* $(\alpha, f)_0=(\alpha, f+g)_0$ *for all* $g\in m^k$.
(b) *If* $\alpha=\alpha_1\cap\alpha_2$ *so that* $\mathbf{dim}_{\mathbf{C}}\,\mathcal{O}/\alpha_1+\alpha_2<\infty$, *then* $(\alpha, f)_0=(\alpha_1, f)_0+(\alpha_2, f)_0$
(c) $(\alpha, f_1\cdot f_2)_0=(\alpha, f_1)_0+(\alpha, f_2)_0$.

If α is an ideal and $f\in\mathcal{O}$, then we shall denote by α_f the ideal generated by α in the ring of fractions of $\mathcal{O}$ with respect to the multiplicative system $\{f^n\}_{n\geqslant 0}$. Let $\gamma_f(\alpha)$ denote the ideal $\alpha_f\cap\mathcal{O}$ in $\mathcal{O}$ and $\sigma_f(\alpha)$ the intersection of the remaining isolated primary components of α.

2.0.2. Let $h, g\in\mathcal{O}$ be such that the pair $(h,g):(\mathbf{C}^{n+1},0)\to(\mathbf{C}^2,0)$ defines an isolated complete intersection singularity (this will be abbreviated in the sequel by ICIS). Recall that the Milnor fiber F_u of $u=(h,g)$ is a bouquet of $\mu_u=\mu(h,g)$ spheres of dimension $n-1$. (If $n=1$, then $\mu(h,g)=(h,g)_0-1$).

Let c be the ideal in $\mathcal{O}$ generated by the 2×2 minors in the jacobian matrix of (h,g). Obviously, the zero-set $V(c)$ of the ideal c is precisely the set of critical points of the map (h,g). Let D be the *reduced* discriminant locus $u(V(c))$ and $D_1,\dots,D_s$ its irreducible components. Let $\Gamma_{i1},\dots,\Gamma_{it_i}$ be the irreducible components of $u^{-1}(D_i)$ $(i=1,\dots,s)$ and d_{ij} the topological degree of the branched covering $\Gamma_{ij}\to D_i$ $(i=1,\dots,s;\,j=1,\dots,t_i)$. If $z\in\Gamma_{ij}-\{0\}$ we denote by $\mu_{ij}(z)$ (resp. $F_{ij}(z)$), the Milnor number (resp. the Milnor fiber) of the ICIS $(h,g):(\mathbf{C}^{n+1},z)\to(\mathbf{C}^2,(h,z)(z))$. Note that $\mu_{ij}(z)$ does not depend on the choice of z on $\Gamma_{ij}-\{0\}$.

Let $\bar{D}_\eta=\{(c,d)\in\mathbf{C}^2:|c|^2+|d|^2\leqslant\eta^2\}$ and $\partial\bar{D}_\eta=S^3_\eta$, $*\in S^3_\eta-D$. For η sufficiently small one defines the *monodromy representation*

$$\rho:\pi_1(\bar{D}_\eta-D,*)\to\mathbf{Aut}(H^{n-1}(F_u,\mathbf{C})).$$

Since $S^3_\eta-D$ is a deformation retract in $\bar{D}_\eta-D$, ρ can be identified with $\pi_1(S^3_\eta-D,*)\to\mathbf{Aut}(H^{n-1}(F_u,\mathbf{C}))$ (still denoted by ρ).

The discriminant locus defines a link in S^3_η with link components

$K_i^D = D_i \cap S_\eta^3$. Set $K_{ij} = (u \mid \Gamma_{ij})^{-1}(K_i^D) \subset \Gamma_{ij}$ for each i and j and consider T_{ij} a small closed tubular neighbourhood of K_{ij}(in $\mathscr{X} \subset \mathbf{C}^{n+1}$, where $(h, g): \mathscr{X} \to \bar{D}_\eta$ is an "excellent representative" of the ICIS [20]). Let $N(K_i^D)$ be a closed tubular neighbourhood of K_i in S_η^3 such that for all $(c, d) \in N(K_i^D)$ the fiber $u^{-1}(c, d)$ meets transversely the boundary ∂T_{ij}. We shall denote by M_i and L_i an oriented "topologically standard" meridian and longitude of the link component K_i^D on $\partial N(K_i^D)$, which are determined up to isotropy by the homology and linking relations

$$
\begin{array}{lll}
M_i \sim 0 & L_i \sim K_i^D & \text{in } H_1(N(K_i^D), \mathbf{Z}) \\
\ell(M_i, K_i^D) = 1 & \ell(L_i, K_i^D) = 0 & (\ell(\,,) = \text{linking number})
\end{array}
$$

If $(c, d) \in K_i^D$, then the singular fiber $u^{-1}(c, d) = F_i^{\mathrm{Sing}}$ has $\Sigma_{j=1}^{t_i} d_{ij}$ singular points. The disjoint union of the corresponding Milnor fibers is $\bigcup_j \bigcup_{d_{ij}} F_{ij} = \bigcup_{j=1}^{t_i} u^{-1}(c, d) \cap T_{ij}$. Therefore we have an action of $\pi_1(\partial N(K_i^D)) = \mathbf{Z} \times \mathbf{Z}$ (generated by the class of M_i and L_i) on $\bigcup_{j=1}^{t_i} \bigcup_{d_{ij}} F_{ij}$ which preserves the subspaces $\bigcup_{d_{ij}} F_{ij} = u^{-1}(c, d) \cap T_{ij}$ for each j. In particular M_i resp. L_i defines the *meridian* resp. the *longitudinal monodromy* $h_{M_i}, h_{L_i}: H^{n-1}(\bigcup_{j=1}^{t_i} \bigcup_{d_{ij}} F_{ij}, \mathbf{C}) \circlearrowleft$ with *decomposition*

$$h_{M_i} = h_{M_i}^{(1)} \oplus \cdots \oplus h_{M_i}^{(t_i)}, \; h_{L_i} = h_{L_i}^{(1)} \oplus \cdots \oplus h_{L_i}^{(t_i)}$$

and after a suitable choice of base of $\bigoplus_{d_{ij}} H^{n-1}(F_{ij}, \mathbf{C})$

$$
h_{M_i}^{(j)} = \begin{bmatrix} M_{ij} & & & 0 \\ & M_{ij} & & \\ & & \ddots & \\ 0 & & & M_{ij} \end{bmatrix}
\qquad
h_{L_i}^{(j)} = \begin{bmatrix} 0 & & & & L_{ij} \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix}
$$

$M_{ij}, L_{ij}: H^{n-1}(F_{ij}, \mathbf{C}) \circlearrowleft$ $(i = 1, \ldots, s; j = 1, \ldots, t_i)$.

Obviously, we have by the commutativity of $\pi_1(\partial N(K_i))$: $M_{ij} L_{ij} = L_{ij} M_{ij}$.

Note that this construction can be regarded as a particular case of the monodromy of local systems [6]. M_{ij} is in fact the horizontal monodromy of a transversal section, L_{ij} is the vertical monodromy of the transversal local system. (See also [33].)

By Thom's first isotopy lemma ([41]), over K_i^D u is a topologically locally trivial map with fiber F_i^{Sing} (which is also a bouquet of $(n-1)$-spheres), thus defining a *singular monodromy* $h_i^{\mathrm{Sing}}: H^{n-1}(F_i^{\mathrm{Sing}}, \mathbf{C}) \circlearrowleft$ $(i = 1, \ldots, s)$. We take $\Delta_i^{\mathrm{Sing}}(\lambda) = \mathbf{det}(1 - \lambda h_i^{\mathrm{Sing}})^{(-1)^n}$. (If $H^{n-1}(F_i^{\mathrm{Sing}}, \mathbf{C}) = 0$ then $\Delta_i^{\mathrm{Sing}}(\lambda) = 1$).

By a Mayer–Vietoris argument for all $a_i \in \mathbf{Z}$ we have

$$\Delta_i^{\mathrm{Sing}}(\lambda) = \left[\frac{\det(1-\lambda\rho(L_i+a_iM_i))}{\det(1-\lambda h_{L_i}\circ h_{M_i}^{a_i})}\right]^{(-1)^n}$$

(in particular the right-hand term is independent of $a_i \in \mathbf{Z}$).

2.0.3. Let $P:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ be an analytic germ in two variables, and let $P=P_1^{m_1}\cdots P_r^{m_r}$ be the prime decomposition of P. Then the Milnor fiber Q_P of P has $\mu_0=\text{g.c.d. }(m_1,\ldots,m_r)$ connected components and each of them has the homotopy type of a bouquet of μ_P circles.

If D is the reduced discriminant locus of the ICIS (h,g) (see (2.0.2)) there exists $P^D:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ such that $V(P^D)=D$. Let $P_1,\ldots,P_t$ be the common components of P and P^D, i.e., $P=P_1^{m_1}\cdots P_t^{m_t}P_{t+1}^{m_{t+1}}\cdots P_r^{m_r}$; $P^D=P_1\cdots P_tP_1^D\cdots P_{s-t}^D$.

2.0.4. Let us define $f\in\mathcal{O}$ by $f=P(h,g)$, where (h,g) is an ICIS and P is as in (2.0.3). We use the notations $\Gamma_f=V(\gamma_f(c))$, $\Sigma_f=V(\sigma_f(c))$. Recall, that in this case $\mathcal{O}/c$ is a 1-dimensional Cohen–Macaulay ring, and in particular $V(c)$ has no embedded components at the origin. Γ_f(resp. Σ_f) consists of the components of $V(c)$ that *are not* (resp. that *are*) contained in $V(f)$. In particular, if D and $V(P)$ have no common irreducible components, then $\Sigma_f=\emptyset$.

2.0.5. If $a_*\in\mathbf{Aut}(H^*(X,\mathbf{C}))$, we define the zeta function of a_* by $\zeta(a_*)(\lambda)=\Pi_q\,\mathbf{det}(1-\lambda a_q)^{(-1)^{q+1}}$. If $f\in\mathcal{O}$, then the zeta function of f is $\zeta_f(\lambda)=\zeta(h_{f,*})(\lambda)$, where $h_{f,*}:H^*(F,\mathbf{C})\circlearrowleft$ is induced by the geometric monodromy $h_f:F\circlearrowleft$.

The following lemma will be very useful:

LEMMA [33]. *Let*

$$M=\begin{bmatrix} 0 & & & & M_k \\ M_1 & & & & \\ & M_2 & & & \\ & & \ddots & & \\ 0 & & & M_{k-1} & 0 \end{bmatrix} \quad \textit{where} \quad M_i\in\mathbf{Mat}_{n\times n}(\mathbf{C})$$

Then $\mathbf{det}(I-\lambda M)=\mathbf{det}(I-\lambda^kM_1\ \ M_2\ \ \cdots\ \ M_k)$.

Some notations:

$B_\varepsilon=\{z\in\mathbf{C}^{n+1}:\|z\|<\varepsilon\}$, $\bar{B}_\varepsilon$ its closure, $\partial\bar{B}_\varepsilon$ its boundary;
$D_\eta=\{(c,d)\in\mathbf{C}^2:\|(c,d)\|<\eta\}$, $\bar{D}_\eta$ its closure, $\partial\bar{D}_\eta$ its boundary;
$D_\delta^1=\{w\in\mathbf{C}:|w|<\delta\}$, $\bar{D}_\delta^1$ its closure, $\partial\bar{D}_\delta^1$ its boundary;
$\langle v^1,v^2\rangle=\Sigma_1^{n+1}v_i^1\bar{v}_i^2$; $v^1,v^2\in\mathbf{C}^{n+1}$.

2.1. The homotopy type of the Milnor fiber

With the above notations we have the following

THEOREM A.

(a) *The Milnor fiber F of $f = P(h,g)$ has the homotopy type of a space obtained from the total space of a fiber bundle with fiber F_u and base space Q_P by attaching $(\gamma_f(c), f)_0$ cells of dimension n.*

In particular the natural map $F \to \bigcup_{\mu_0} K(\pi_1(F), 1)$ is an $(n-1)$-equivalence (the latter space is a disjoint union of μ_0 copies of the Eilenberg–MacLane space), and the Euler-characteristic of F is:

$$\begin{aligned}\chi(F) &= \chi(Q_P)\cdot\chi(F_u) + (-1)^n(\gamma_f(c), f)_0 \\ &= (1-\mu_P)\mu_0 + (-1)^n[(\mu_P - 1)\mu_u\mu_0 + (\gamma_f(c), f)_0].\end{aligned}$$

(b) *If $\Sigma_f = \varnothing$, F has the homotopy type of μ_0 copies of a bouquet of spheres $(\bigvee_{\mu_1} S^1) \vee (\bigvee_{\mu_n} S^n)$, where*

$$\mu_1 = \mu_P, \qquad \mu_n = (\mu_P - 1)\mu_u + (c, f)_0/\mu_0$$

Moreover, the protection $\bigcup_{\mu_0}(\bigvee_{\mu_1} S^1 \vee \bigvee_{\mu_n} S^n) \to \bigcup_{\mu_0}\bigvee_{\mu_1} S^1$ can be identified to the map $u\colon F \to Q_P$.

The proof of Theorem A will be given in (3.2).

2.1.1. COROLLARY. *Let*

$$\varepsilon(f) = \begin{cases} 0 & \text{if } \Sigma_f = \varnothing \\ 1 & \text{if } \Sigma_f \neq \varnothing \end{cases} \quad \text{and suppose that } \mu_0 = 1.$$

If $n \geqslant 2 + \varepsilon(f)$, then $\pi_1(u)\colon \pi_1(F) \to \pi_1(Q_P)$ is an isomorphism.

Theorem A contains various well known formulae as particular cases.

EXAMPLES

2.1.2. Let (h, g) be as in Theorem A, with $h \in \mathcal{O}$ an IS (isolated singularity). Assume that $P(c, d) = c$; in particular $f = h$. Since h is an IS, $\Sigma_f = \varnothing$. Therefore our formula in this case becomes:

$$\mu(f) + \mu(f, g) = (c, f)_0$$

(the Milnor-number formula of the ICIS, see for example [20] p. 77).

If, moreover, $g^{-1}(0)$ is a generic hyperplane, then we get Teissier's ([36, p. 317]) formula

$$\mu^{(n+1)}(f) + \mu^{(n)}(f) = (c, f)_0$$

2.1.3. Let $h \in \mathcal{O}$ be such that **dim Sing** $V(h) = 1$, and assume that (h, g) is an ICIS (the latter condition is automatically fulfilled if g is a generic linear form). Assume again that $P(c, d) = c$. Then the Milnor fiber F_h of h is $(n-2)$-connected and we have the following relation between the Betti numbers:

$$b_n(F_h) - b_{n-1}(F_h) + \mu(h, g) = (\gamma_h(c), h)_0$$

If $V(g)$ is a generic hyperplane, we get the "Lê attaching formula" [16] (for germs with **dim Sing** $V(h) = 1$).

2.1.4. Let (h, g) be an ICIS and take $P(c, d) = c + d^k$ $(k \geqslant 1)$, i.e. $f = h + g^k$. Again $F_{h+g^k} = F_u \cup \{(\gamma_f, h + g^k)_0 \text{ cells of } \mathbf{dim}\, n\}$. Since the curves $V(c + d^k)$ $(k \geqslant 1)$ have no irreducible components in common, there exists k_0 such that if $k \geqslant k_0$, D and $V(c + d^k)$ have no irreducible components in common, in particular $\Sigma_f = \emptyset$ and $c = \gamma_f(c)$. In this case $h + g^k$ is an IS. By Lemma (2.0.1) and Theorem A:

$$\mu(h + g^k) + \mu(h, g) = (\gamma_h(c), h)_0 + k(\sigma_h(c), g)_0 \quad \text{if } k \gg 0.$$

Obviously **dim Sing** $V(h) \leqslant 1$, hence by (2.1.3) we get:

$$\mu(h + g^k) = b_n(h) - b_{n-1}(h) + k(\sigma_h(c), g)_0 \quad \text{if } k \gg 0.$$

If g is generic linear form, then this is exactly the Iomdin–Lê formula [14, 18].

2.1.5. Let (h, g) be an ICIS such that h and g are IS. Then for $P_{k,l}(c, d) = c^k d^l$ $(k \geqslant 1, l \geqslant 1)$, $\Sigma_f = \emptyset$. The numerical invariants of the bouquet are: $\mu_0 = \text{g.c.d.}(k, l)$, $\mu_1 = \mu_P = 1$ and (using (2.1.1))

$$\mu_n = (\mu(h, g) + \mu(h)) \cdot k/\mu_0 + (\mu(h, g) + \mu(g)) \cdot l/\mu_0.$$

2.1.6. Let $h(z) = z_{n+1}$ and $g(z) = \varphi(z_1, \ldots, z_n)$, where $\varphi: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ defines an IS. Then by a theorem of Sakamoto [25] the Milnor fiber F of $f = h^k g$ may be regarded as the total space of a fiber bundle over S^1 with fiber the Milnor fiber of φ (and monodromy m_φ^k). Thus if $n > 2$, $\pi_{n-1}(F) = \mathbf{Z}^{\mu(\varphi)}$. Hence our assumption in Theorem A(b) is necessary.

2.1.7. Let $(f, \emptyset)$ be an ICIS such that $f + \emptyset$ is an IS. Assume that the order of $\emptyset$ is greater than $\mathbf{dim}_{\mathbf{C}}\, \mathcal{O}/\gamma_f(c) + (f)$. Then

$$\mu(f + \emptyset) = (-1)^n(\chi(f) - 1) + (\sigma_f(c), \emptyset)_0$$

This corollary gives an (partial) answer to a question raised by Wall, as reformulated by Siersma [23, p. 115].

2.2. *The zeta function of the monodromy*

In this section we determine the zeta function of the monodromy of the germ $f = P(h,g)$. If $P = (P')^{\mu_0}$ and $f' = P'(h,g)$, then we have the following relations between the corresponding Milnor fibers and zeta functions:

$$F_f = F_{f'} \times \mathbf{Z}_{\mu_0};\ \zeta_f(\lambda) = \zeta_{f'}(\lambda^{\mu_0}) \quad \text{(by 2.0.5)}.$$

Therefore, we may assume (with no loss of generality) that F_f is connected, i.e., g.c.d. $(m_1, \ldots, m_r) = 1$.

2.2.1. LEMMA. *If η is sufficiently small, then*

$$\varphi_P = \frac{P}{|P|}: (S^3_\eta - P^{-1}(0),\ S^3_\eta \cap D - P^{-1}(0)) \to S^1$$

$$P: (P^{-1}(\partial \bar{D}^1_\delta) \cap \bar{D}_\eta,\ P^{-1}(\partial \bar{D}^1_\delta) \cap \bar{D}_\eta \cap D) \to \partial \bar{D}^1_\delta \quad (0 < \delta \ll \eta)$$

are locally trivial fibrations of pairs of spaces. Moreover, the fibrations are fiber isomorphic.

Proof. This is a consequence of the fact that for η small, the fibers $\varphi_P^{-1}(e^{2\pi i\theta})$ in $S^3_\eta - P^{-1}(0)$ (resp. $P^{-1}(w)$, $w \in \partial \bar{D}^1_\delta$) meet transversely the link components $S^3_\eta \cap D$ (resp. D) (to see this use the Curve Selection lemma [20]). In order to prove the isomorphism we construct a vector field similarly as in the non-relative case [21, pp. 52–53].

2.2.2. REMARK. A fiber of φ_P can be identified with a minimal Seifert surface of the multilink corresponding to the link:

$$L = (P_1^{-1}(0) \cap S^3_\eta, \ldots, P_r^{-1}(0) \cap S^3_\eta, (P_1^D)^{-1}(0) \cap S^3_\eta, \ldots, (P^D_{s-t})^{-1}(0) \cap S^3_\eta)$$

with multiplicities $(m_1, \ldots, m_r, 0, \ldots, 0)$ ([9], *p.* 34).

In fact (2.2.1) also follows from the fact that $L(m,0)$ is a fibered multilink.

Let $*$ be a point of $F_\varphi - D (F_\varphi = \varphi_P^{-1}(1))$, then by (2.2.1) we have the exact sequence of groups:

$$\text{(2.2.3)} \quad 1 \to \pi_1(F_\varphi - D, *) \xrightarrow{i_*} \pi_1(S^3_\eta - P^{-1}(0) \cup D, *) \xrightarrow{\varphi_*} \mathbf{Z} \to 1$$

By the inclusion $S^3_\eta - P^{-1}(0) \cup D \hookrightarrow S^3_\eta - D$, $A = H^{n-1}(F_u, \mathbf{C})$ becomes a $G = \pi_1(S^3_\eta - P^{-1}(0) \cup D, *)$-module (induced by ρ), and by i_* a $H = \pi_1(F_\varphi - D, *)$-module.

Let $g \in G$ such that $\varphi_*(g) = 1$. Then the maps $\rho_g = \rho(g)\colon A \to A$ and $c_g\colon H \to H$, $h \to g^{-1} \cdot h \cdot g$ induce an automorphism of the exact sequence (of **C**-vector spaces):

(2.2.4)

$$\begin{array}{ccccccccc} 0 \to & H^0(H, A) & \to & A & \xrightarrow{\delta} & \mathrm{Der}(H, A) & \to & H^1(H, A) & \to 0 \\ & \downarrow g_0^* & & \downarrow \rho_g & & \downarrow g_{\mathrm{Der}} & & \downarrow g_1^* & \\ 0 \to & H^0(H, A) & \to & A & \xrightarrow{\delta} & \mathrm{Der}(H, A) & \to & H^1(H, A) & \to 0 \end{array}$$

(where $\delta(a)(h) = \rho_h \cdot a - a$).

Let us define

$$(\zeta_{P,D}(\lambda))^{(-1)^n} = \mathbf{det}(1 - \lambda g_0^*)/\mathbf{det}(1 - \lambda g_1^*)$$
$$= \Delta_g(\lambda)/\Delta_{\mathrm{Der}}(\lambda)$$

where

$$\Delta_g(\lambda) = \mathbf{det}(1 - \lambda\rho_g) \quad \text{and} \quad \Delta_{\mathrm{Der}}(\lambda) = \mathbf{det}(1 - \lambda g_{\mathrm{Der}}).$$

By [27, p. 116] the automorphisms g_0^* and g_1^* do not depend on the choice of g. Hence $\zeta_{P,D}$ is well defined by the representation ρ and the exact sequence (2.2.3).

2.2.5. REMARK. When $\Sigma_f = \varnothing$ then g can be chosen such that the image of g in $\pi_1(S^3_\eta - D, *)$ is trivial, hence $\rho_g = 1_A$.

The zeta function of P is $\zeta_P(\lambda) = \Delta_P(\lambda)/(1 - \lambda)$, where $\Delta_P(\lambda)$ is the characteristic polynomial of the monodromy of P (on $H^1(Q_P, \mathbf{C})$). With this notation we have the following

THEOREM B. *Let $f = P(h, g)$ be as in Theorem A. Then* (a)

$$\zeta_f(\lambda) = \zeta_P(\lambda)\varepsilon \cdot \zeta_{P,D}(\lambda) \cdot \prod_{j=1}^{s-t} \Delta_j^{\mathrm{Sing}}(\lambda^{k_j}) \quad (\varepsilon = 0 \text{ if } n = 1 \text{ and } \varepsilon = 1 \text{ if } n \geqslant 2)$$

where $k_j = (P, P_j^D)_0 (= \#(Q_P \cap D_j))$ (and the product is taken over the components of D which are not contained in $P^{-1}(0)$).

(b) *Suppose $\Sigma_f = \varnothing$ and $n \geqslant 2$. Then*

$$\mathbf{det}(1 - \lambda h_{f,q}) = \begin{cases} 1 - \lambda & \textit{if } q = 0 \\ \Delta_P(\lambda) & \textit{if } q = 1 \\ \Delta_{\mathrm{Der}}(\lambda) \cdot \Delta_g(\lambda)^{-1} \cdot \prod_{j=1}^{s-t} (\Delta_j^{\mathrm{Sing}}(\lambda^{k_j}))^{(-1)^{n+1}} & \textit{if } q = n \end{cases}$$

(the product is taken over the all components of D).

The proof will be given in (3.3).

2.2.6. REMARK. Since H is a free group with $\mu_P + \Sigma_{i=1}^{s-t} k_i = \mu(P, D)$ generators, say $b_1, \ldots, b_{\mu(P,D)}$, $\mathrm{Der}(H, A)$ can be identified with $A^{\mu(P,D)}$ by $\delta \to (\delta(b_1), \ldots, \delta(b_{\mu(P,D)}))$. The corresponding matrix $[g_{\mathrm{Der}}]: A^{\mu(P,D)} \to A^{\mu(P,D)}$ may be regarded as the "Jacobian matrix of (G, H) at ρ" [10]. More precisely:

Let us consider the following homomorphisms of rings:

$\tilde{i}_*: \mathbf{Z}[H] \to \mathbf{Z}[G]$ induced by i_*

$\tilde{\rho}: \mathbf{Z}[G] \to \mathbf{Z}[\mathrm{Aut}\, A]$ induced by ρ

$s: \mathbf{Z}[\mathrm{Aut}\, A] \to \mathbf{End}_{\mathbf{C}} A$ given by $s(\Sigma_i c_i [a_{jk}^i]_{jk}) = [\Sigma_i c_i a_{jk}^i]_{jk}$

and the derivations

$$\frac{\partial}{\partial b_j}: \mathbf{Z}[H] \to \mathbf{Z}[H] \text{ determined by } \frac{\partial b_i}{\partial b_j} = \delta_{ij} \quad \text{(for each } i \text{ and } j\text{)}$$

(where $\mathbf{Z}[G]$ denotes the group ring of a group G).

We denote the words $c_g(b_i)$ by w_i. Then using the derivation rule we obtain that $[g_{\mathrm{Der}}]$ is in fact the matrix constructed by block-matrices

$$\left[.s \circ \tilde{\rho} \left(g \cdot \tilde{i}_* \left(\frac{\partial w_i}{\partial b_j} \right) \right) \right]_{ij}.$$

Therefore, when we know explicitly the action $c_g: H \to H$ (i.e. the extension (2.2.3)), and the representation ρ, the $\zeta_{P,D}$ follows explicitly from the above identification.

2.2.7. The above considerations can be made more precise if *imρ is an abelian group*, because in this case the Jacobian matrix of (G, H) at ρ may be factorized by the Alexander matrix of (G, H).

Suppose that we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(S_\eta^3 - D, *) & \xrightarrow{\ \rho\ } & \mathbf{Aut}\, A \\ & \searrow^{\chi} \quad \nearrow_{\rho_{ab}} & \\ & \pi_1(.S_\eta^3 - D, *)_{ab} & \end{array}$$

where χ is the abelianizer. Obviously, in this case we also have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathbf{Aut}\, A \\ & \chi \searrow & \nearrow \rho_{ab} \\ & G_{ab} & \end{array}$$

We can choose the base elements $e_i=(0,\ldots,1,\ldots,0)$ of $G_{ab}=\mathbf{Z}^{r+s-t}$ by taking topological standard meridians in tubular neighbourhoods of the link components of L (see (2.2.2)). For each $i=1,\ldots,r+s-t$ we have an automorphism $E_i=\rho_{ab}(e_i)\in \mathbf{Aut}\, A$ $(E_{t+1}=\cdots=E_r=1)$, and multiplicities $m_i(m_{r+1}=\cdots=m_{r+s-t}=0)$. The ring group $\mathbf{Z}[G_{ab}]$ is $\mathbf{Z}[\lambda_1,\lambda_1^{-1},\ldots,\lambda_{r+s-t},\lambda_{r+s-t}^{-1}]$ where λ_i correspond to e_i.

Let $\Delta(\lambda_1,\ldots,\lambda_{r+s-t})$ (resp. $\Delta(\lambda)$) be the Alexander polynomial of the link L if $r+s-t\geqslant 2$ (resp. $r+s-t=1$) in $\mathbf{Z}[G_{ab}]$.

THEOREM C. *If* $\operatorname{im}\rho$ *is an abelian group, then*

$$(\zeta_{P,D}(\lambda))^{(-1)^{n+1}}=\begin{cases} \mathbf{det}\,\Delta(\lambda^{m_1}E_1,\ldots,\lambda^{m_{r+s-t}}E_{r+s-t} & \text{if } r+s-t\geqslant 2 \\ \dfrac{\mathbf{det}\,\Delta(\lambda^{m_1}E_1)}{\mathbf{det}(I-\lambda^{m_1}E_1)} & \text{if } r+s-t=1 \end{cases}$$

We recall that Δ *is well-defined only up to multiplication by monomials* $\pm\lambda_1^{i_1}\cdots\lambda_{r+s-t}^{i_{r+s-t}}$, *therefore the above equality is to be taken modulo these monomials.*

The proof is given in (3.4).

2.2.8. REMARK. A particular case when $\operatorname{im}\rho$ is abelian occurs when $\pi_1(S_\eta^3-D,*)$ itself is abelian. This happens exactly when either D is smooth or D is singular of type A_1.

2.2.9. REMARK. Consider the case when D is smooth, $V(c)$ is irreducible and the Milnor number $\mu(z)$ of the ICIS $(h,g)\colon(\mathbf{C}^{n+1},z)\to(\mathbf{C}^2,(h,g)(z))$ is constant for $z\in V(c)$ (in a neighbourhood of the origin). Then F_1^{Sing} is homeomorphic to the central singular fiber, hence it is contractible, thus $\Delta_1^{\text{Sing}}(\lambda)=1$. Therefore

$$\zeta_f(\lambda)=\zeta_P(\lambda)\cdot\mathbf{det}\,\Delta_*(\lambda^{m_1}E_1,\ldots,\lambda^{m_{r+s-t}}E_{r+s-t})^{(-1)^{n+1}}$$

i.e.

$$\mathbf{det}(1-\lambda h_{f,n})=\mathbf{det}\,\Delta_*(\lambda^{m_1}E_1,\ldots,\lambda^{m_{r+s-t}}E_{r+s-t})$$

where

$$\Delta_*(\lambda_1,\dots,\lambda_{r+s-t}) = \begin{cases}\Delta(\lambda_1,\dots,\lambda_{r+s-t}) & \text{if } r+s-t \geqslant 2\\ \Delta(\lambda_1)/(1-\lambda_1) & \text{if } r+s-t=1\end{cases}$$

2.2.10. EXAMPLE. Take $h = z_{n+1}$ and $g(z) = \tilde{g}(z_1,\dots,z_n)$, where $\tilde{g}:(\mathbf{C}^n,0)\to(\mathbf{C},0)$ defines an IS. Then $D=\{d=0\}$. Let $P=P_1^{m_1}\cdots P_r^{m_r}$ such that $\Sigma_f=\varnothing$. Then

$$\Delta_{f,n}(\lambda) = \mathbf{det}(1-\lambda h_{f,n}) = \mathbf{det}\,\Delta(\lambda^{m_1}I,\dots,\lambda^{m_r}\cdot I,\, h_{\tilde{g}})$$

where $h_{\tilde{g}}: H^{n-1}(F_{\tilde{g}},\mathbf{C}) \circlearrowleft$ is the algebraic monodromy of $\tilde{g}$. For example if $f = z_{n+1}^k + \tilde{g}(z_1,\dots,z_n)$, then

$$\Delta_{f,n}(\lambda) = \mathbf{det}(1-\lambda^k h_{\tilde{g}}^k)/\mathbf{det}(1-\lambda h_{\tilde{g}}) \quad \text{if } f = z_{n+1}^k + z_{n+1}\tilde{g},$$

then $\Delta_{f,n}(\lambda) = \mathbf{det}(1-\lambda^k h_{\tilde{g}}^{k-1})$.

We note that Δ_* can be immediately calculated using the EN-diagram [9].

2.3. Series of singularities

2.3.1. Let $(h,g):(\mathbf{C}^{n+1},0)\to(\mathbf{C}^2,0)$ be an ICIS with reduced discriminant locus D, $P_\infty:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ a germ with two variables and $f_\infty = P_\infty(h,g)$. Then by Theorem A

$$\chi(f_\infty) = \chi(P_\infty)\cdot\chi(F_u) + (-1)^n(\gamma_{f_\infty}(c), f_\infty)_0.$$

Let $\{P_k\}_{k\geqslant k_0}$ be a series of plane curve singularities, such that

(i) $P_k^{-1}(0)$ and D have no common components

(ii) $\lim_{k\to\infty}\mathbf{mult}_0(P_\infty - P_k) = \infty$ (where $\mathbf{mult}_0(P)$ denotes the multiplicity of P).

By Theorem A we get for $f_k = P_k(h,g)$

$\chi(f_k) = \chi(P_k)\cdot\chi(F_u) + (-1)^n(c, f_k)_0$ and by (2.0.1)

(2.3.2) $\chi(f_k) - \chi(f_\infty) = [\chi(P_k) - \chi(P_\infty)]\chi(F_u) + (-1)^n(\sigma_{f_\infty}(c), f_k)_0$

2.3.3. EXAMPLE. Consider the following series of singularities

$W^{\#}_{1,\infty}$	$P_\infty = (d^2-c^3)^2$	with $\chi(P_\infty) = -2$
$W^{\#}_{1,2q-1}$	$P_{2q-1} = (d^2-c^3)^2 + c^{4+q}\cdot d$	with $\chi(P_{2q-1}) = -13-2q$
$W^{\#}_{1,2q}$	$P_{2q} = (d^2-c^3)^2 + c^{3+q}\cdot d$	with $\chi(P_{2q}) = -14-2q$

Then

$$\chi(f_{2q-1})-\chi(f_\infty)=-(11+2q)\chi(F_u)+(-1)^n[(4+q)(\sigma_{f_\infty},h)_0+(\sigma_{f_\infty},g)_0]$$

$$\chi(f_{2q})-\chi(f_\infty)=-(12+2q)\chi(F_u)+(-1)^n[(3+q)(\sigma_{f_\infty},h)_0+(\sigma_{f_\infty},g)_0].$$

2.3.4. Series of singularities appear naturally in the classification of germs. The Arnold's series.of isolated singularities are related to non-isolated singularities, by "approximating" them. This is the main motivation for Schrauwen's [26] definition of topological series of isolated plane curve singularities: the topological series belonging to P_∞ consists of all topological types of isolated singularities whose links arise as the splice of the link of P_∞ with some other links.

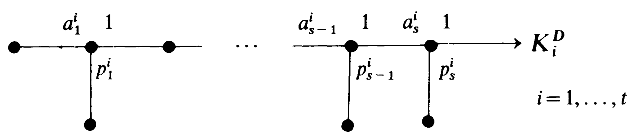
Similarly, in our case, when $P_\infty^{-1}(0)$ and D do have common irreducible components, we want to approximate the germ f_∞ by germs $f_k=P_k(h,g)$ such that f_k is a "nice germ" in our class, i.e. $\Sigma_{f_k}=\emptyset$. The best approximation of f_∞ with f_k holds when $\chi(P_k)=\chi(P_\infty)$ (see formula (2.3.2)), i.e. when the topological type of P_∞ and P_k agree, and only the mutual position of $P_\infty^{-1}(0)$ and $P_k^{-1}(0)$ with respect to D varies.

Recall the notations: $P_\infty=P_1^{m_1}\cdots P_t^{m_t}\cdot P_{t+1}^{m_{t+1}}\cdots P_r^{m_r}$ and D is given by $P^D=P_1\cdots P_tP_1^D\cdots P_{s-t}^D$.

The link $K_i^D=S_\eta^3\cap P_i^{-1}(0)$ $(i=1,\dots,t)$ is given by the expansion

$$P_i\colon y=x^{q_1^i/p_1^i}(c_1+\cdots(c_{s-1}+c_sx^{q_s^i/p_1^i\cdots p_s^i})\cdots)$$

We represent it by the EN-diagram:

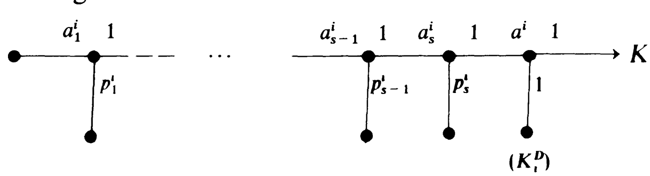


where a_j^i are given from the Newton pairs $(p_j^i,q_j^i)_j$ by the formula $a_1^i=q_1^i$, $a_{j+1}^i=q_{j+1}^i+p_j^ip_{j+1}^ia_j^i$ if $j\geqslant 2$.

Consider the following developments for $i=1,\dots,t$:

$$P(q^i)\colon y^1=x^{q_1^i/p_1^i}(c_1+\cdots(c_{s-1}+x^{q_s^i/p_1^i\cdots p_s^i}(c_s+x^{q^i/p_1^i\cdots p_s^i})\cdots)$$

with EN-diagram



where $a^i=q^i+p_s^ia_s^i$.

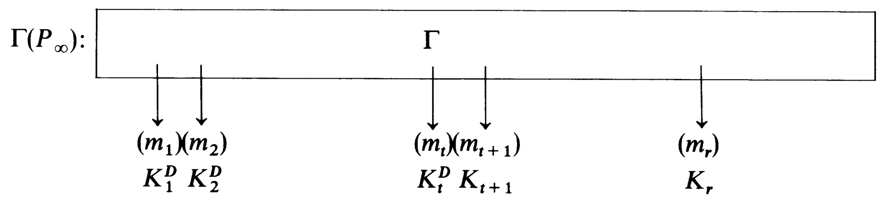
Thus, the link component $K_i^D (i=1,\dots,t)$ is replaced by K_i', a cable on K_i^D, which will follow K_i^D around 1-times in the longitudinal direction and a^i-times in the meridian direction.

2.3.5. DEFINITION. For all $\{a^i\}_{i=1,\dots,t}$, with $a^i > p_s^i a_s^i$, we define $P_{(a^1,\dots,a^t)} = P_{\mathbf{a}} = P(q^1)^{m_1} \cdots P(q^t)^{m_t} P_{t+1}^{m_{t+1}} \cdots P_r^{m_r}$ and we say that $P_{\mathbf{a}}$ is a "topological trivial series" belonging to P_∞.

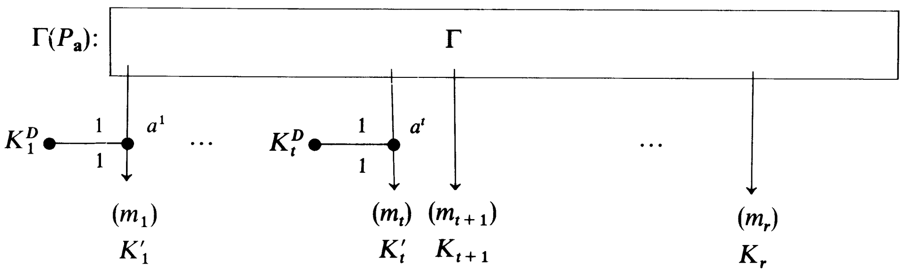
If the schematic EN-diagram of the multilink

$$(S^3, m_1 P_1^{-1}(0) \cap S_\eta^3 \cup \cdots \cup m_r P_r^{-1}(0) \cap S_\eta^3)$$

is the following:

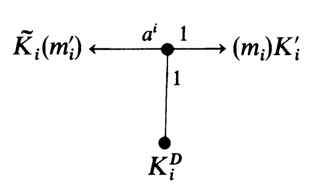


then the schematic EN-diagram of $(S^3, m_1 K_1' \cup \cdots \cup m_t K_t' \cup \cdots \cup m_r K_r)$ is:



Thus $K_1^D, \dots, K_t^D$ remains as "virtual components" of $\Gamma(P_{\mathbf{a}})$, $P_{\mathbf{a}}$ and D have no common components and $\chi(P_{\mathbf{a}}) = \chi(P_\infty)$.

$\Gamma(P_{\mathbf{a}})$ is obtained from $\Gamma(P_\infty)$ by splicing with t (trivial) torus links:



$i = 1, \dots, t$

By splice condition $m_i' = \Sigma_{1 \leqslant i \leqslant r, l \neq j} m_j \ell(K_i, K_j)$ (hence it is an invariant of $\Gamma(P_\infty)$ and by an easy computation it may be calculated from the EN-diagram, see p. 84 [9]). Obviously, the algebraicity condition is fulfilled if a^i is sufficiently large ($a^i > p_s^i a_s^i$).

With this notation we have the following

THEOREM D. *If* $\mathbf{a} \gg \mathbf{0}$ *then for* $f_{\mathbf{a}} = P_{\mathbf{a}}(h, g)$:

$$\zeta_{f_{\mathbf{a}}}(\lambda) = \dot{\zeta}_{f_\infty}(\lambda) \cdot \prod_{i=1}^{t} (\mathbf{det}(1 - \lambda^{m_i' + m_i a^i} h_{M_i}^{a^i} h_{L_i}))^{(-1)^{n+1}}$$

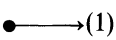
$$= \zeta_{f_\infty}(\lambda) \cdot \left(\prod_{i=1}^{t} \prod_{j=1}^{t_i} \mathbf{det}(1 - \lambda^{d_{ij}(m_i' + m_i a^i)} M_{ij}^{d_{ij} a^i} L_{ij})\right)^{(-1)^{n+1}}$$

In particular

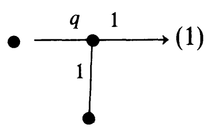
$$\chi(f_a) - \chi(f_\infty) = (-1)^{n+1} \sum_{i=1}^{t} (m_i' + m_i a^i) \sum_{j=1}^{t_i} d_{ij} \mu_{ij}$$

The proof is given in (3.5).

2.3.6. EXAMPLE (Iomdin's series)



$P_\infty(c, d) = c$



$P_q(c, d) = c + d^q$

($t = 1$, $m_1 = 1$, $m_1' = 0$, $a^1 = q$).

Let (h, g) be an ICIS such that dim $\text{Singh}^{-1}(0) = 1$. Thus $\{c = 0\} \subset D$. The series $f_q = h + g^q$ approximate the "bad" singularity $f_\infty = h$.

If g is a general linear form, then we obtain Sierma's result [33].

2.4. *The complement of a projective hypersurface*

2.4.1. Let h, g be homogeneous polynomials in $\mathbf{C}^{n+1}$ ($n \geqslant 2$) of degree $d_h \geqslant 1$ and $d_g \geqslant 1$, such that (h, g) is an ICIS. Let $d_1 = d_h/\text{g.c.d.}(d_h, d_g)$, $d_2 = d_g/\text{g.c.d.}$ (d_h, d_g). Let P be a weighted homogeneous polynomial in two variables with weights d_1 and d_2 and degree d. Then $f = P(h, g)$ is a homogeneous polynomial with degree $d_f = d \cdot (d_h, d_g)$.

Since we want to study the complement space

$$\mathbf{P}^n - [V(f)] =$$
$$\mathbf{P}^n - [V(P_{\text{red}}(h, g))],$$

we may assume in this section, that $P = P_{\mathrm{red}} = P_1 \cdots P_r$.

It is easy to see that the Milnor fiber $F = f^{-1}(1)$ is the total space of a $\mathbf{Z}_{d_f}$-covering over the complement $X = \mathbf{P}^n - [V(f)]$. Therefore, by Theorem A we have:

2.4.2. The natural map $X \to K(\pi_1(X, *), 1)$ is an $(n - \varepsilon(f))$-equivalence. (See 2.1.1 for the definition of $\varepsilon(f)$.)

2.4.3. The aim of this section is to determine the group $\pi_1(X, *)$.

The fiber bundle $E = \mathbf{C}^{n+1} - V(f) \xrightarrow{[\,]} \mathbf{P}^n - [V(f)]$ with fiber $\mathbf{C}^*$ has the homotopy exact sequence

$$1 \to \pi_1(\mathbf{C}^*) \xrightarrow{i_*} \pi_1(E) \to \pi_1(X) \to 1$$

The exact sequence of the fiber bundles $E \xrightarrow{f} \mathbf{C}^*$ with fiber F and $E_P = \mathbf{C}^2 - V(P) \xrightarrow{P} \mathbf{C}^*$ with fiber Q_P are related by the diagram:

$$\begin{array}{ccccccccc} 1 \to & \pi_1(F) & \to & \pi_1(E) & \to & \pi_1(\mathbf{C}^*) & \to 1 \\ & \downarrow u_*^1 & & \downarrow u_*^2 & & \| & \\ 1 \to & \pi_1(Q_P) & \to & \pi_1(E_P) & \to & \pi_1(\mathbf{C}^*) & \to 1 \end{array}$$

Since u_*^1 is an isomorphism for $n \geqslant 2 + \varepsilon(f)$ (by (2.1.1)) we obtain that u_*^2 is an isomorphism. Consequently $\pi_1(X, *) = \pi_1(E_p, *)/N'_*$, where N'_* is the normal subgroup generated by the class of the loop $t \mapsto (h(e^{2\pi i t} \cdot *),\ g(e^{2\pi i t} \cdot *)) = (e^{2\pi i d_h t} h(*),\ e^{2\pi i d_g t} g(*))$, $t \in [0, 1]$.

2.4.4. The complement space $\{(c, d) : |c|^2 + |d|^2 = 1\} - V(P)$ can be identified with $E(P) = \Sigma(d_2, d_1) - \bigcup S_i$, where $\Sigma(d_2, d_1)$ is the Seifert manifold with Seifert invariants d_2 and d_1 [9], and singular orbits S_1 and S_2 (corresponding to $c = 0$ and $d = 0$), and $\bigcup S_i$ is a disjoint union of S^1-orbits such that

(i) $S_1 \subset \bigcup S_i \Leftrightarrow c$ is a component of P
(ii) $S_2 \subset \bigcup S_i \Leftrightarrow d$ is a component of P
(iii) $\bigcup S_i$ contains r' general orbits $\Leftrightarrow P$ has r' irreducible factors different from c and d.

Let $\pi_1(E(P), *)$ with a base point $*$ on a general S^1-orbit S_*, and let N_* be the normal subgroup generated by the class of the loop which follows the S^1-orbit S_* in (d_h, d_g)-times. Then $\pi_1(E(P), _*)/N_* = \pi_1(E_p, _*)/N'_*$ by the above identification.

Our results can be summarized in the following

2.4.5. PROPOSITION. *If $n \geqslant 2 + \varepsilon(f)$, then there exists a $(n - \varepsilon(f))$-equivalence $X \to K(\pi_1(E(P), _*)/N_*, 1)$. In particular $\pi_1(X)$ depends only on the numbers (d_h, d_g) and r', and on the fact that c resp. d is or is not a component of P.*

This result can be regarded as a generalization of Proposition 4.2 of Kato and Matsumoto [15].

2.4.6. EXAMPLES

(a) $P = c, \qquad \pi_1(X) = \mathbf{Z}/(d_h)$

$P = c + d^q, \qquad \pi_1(X) = \mathbf{Z}/(d_h) = \mathbf{Z}/(d_f)$

$P = c \cdot d, \qquad \pi_1(X) = \mathbf{Z}^2/(d_h, d_g)\mathbf{Z}$

These cases may be compared to a general result due to Lê and Saito [19] which says that $\pi_1(X)$ is abelian (hence $\pi_1(X) = H_1(X, \mathbf{Z})$) provided f has normal crossing in codimension 1.

(b) $P = c^3 + d^2, \qquad \pi_1(E_p, *) = \langle x, y \colon xyxy^{-1}x^{-1}y^{-1} \rangle$

$\pi_1(X) = \langle x, y \colon xyxy^{-1}x^{-1}y^{-1} \rangle / \langle (xyxyxy)^{(d_h, d_g)} \rangle$ and by substitution $a = xyx$, $b = xy$ $\pi_1(X) = \langle a, b \colon a^2 = b^3, b^{3(d_h, d_g)} = 1 \rangle$.

In particular if we take $(h, g) = (x^2 + y^2, x^3 + z^3)$ then we obtain the Zariski example $f = (x^2 + y^2)^3 + (x^3 + z^3)^2$ with $\pi_1 = \mathbf{Z}_2 * \mathbf{Z}_3$ [40].

2.4.7. Suppose that $\Sigma_f = \emptyset$, hence $\varepsilon(f) = 0$. The homotopy groups and the Euler characteristic of X can be determined by the $\mathbf{Z}_{d_f}$-covering, and the homology groups $H_q(X, \mathbf{Z})$ $0 \leqslant q \leqslant n - 1$ by the n-equivalence (2.4.5). Since X is a n-dimensional Stein space, the only non-trivial remaining group, $H_n(X, \mathbf{Z})$, is free and may be determined by an Euler-characteristic argument.

2.5. *Germs with finite $(h)^{k-1}$-codimension*

2.5.1. Recall that $\mathcal{O}$ is the local ring of germs of analytic functions of $(n+1)$-variables and m is its maximal ideal. For an analytic germ we denote by J_f its Jacobian ideal, i.e. $J_f = (\partial_1 f, \dots, \partial_{n+1} f)\mathcal{O}$. If I is an ideal in $\mathcal{O}$, then its primitive ideal $\int I$ is defined as $\{f \in \mathcal{O} \mid (f) + J_f \subset I\}$ [23]. Let $\mathcal{D}$ be the group of germs of local analytic isomorphisms $h \colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ and consider the subgroup $\mathcal{D}_I = \{h \in \mathcal{D} \mid h^*(I) = I\}$. Since $\mathcal{D}_I$ acts on $\int I$, for germs $f \in \int I$ we can define the tangent space $\tau_I(f) = \{\xi(f) \mid \xi \in TD_I\}$ (where $TD_I = \{\xi \in m \operatorname{Der}_{\mathbf{C}} \colon \xi(I) \subset I\}$) of the orbit of f under $\mathcal{D}_I$. Similarly, the sub pseudo group $\mathcal{D}_{I,e}$ of the pseudo group $\mathcal{D}_e$ of local analytic isomorphisms $h \colon (\mathbf{C}^{n+1}, 0) \to \mathbf{C}^{n+1}$ determines the extended tangent space $\tau_{I,e}(f) = \{\xi(f) \mid \xi \in TD_{I,e}\}$ (where $TD_{I,e} = \{\xi \in \operatorname{Der}_{\mathbf{C}} \mathcal{O} \mid \xi(I) \subset I\}$) of f.

The I-codimension, resp. the extended I-codimension of f is defined by

$$c_I(f) = \operatorname{\mathbf{dim}}_{\mathbf{C}} \frac{\int I}{\tau_I(f)} \qquad c_{I,e}(f) = \operatorname{\mathbf{dim}}_{\mathbf{C}} \frac{\int I}{\tau_{I,e}(f)}$$

By [23, p. 32] $c_I(f) < \infty$ if and only if $c_{I,e}(f) < \infty$.

2.5.2. Consider $h \in \mathfrak{m}$ with an *isolated critical point* at the origin and let $I = (h)^{k-1} (k \geqslant 2)$. Then by an easy computation we obtain that $\int I = (h)^k$, thus if $f \in \int I$ then $f = h^k \cdot g$. It is easy to see, that if g has only isolated singularities then the critical locus of f is $V(h)$.

The purpose of this section is to determine the topological type of the Milnor fiber of germs $f = h^k \cdot g$ with finite I-codimension.

2.5.3. If $g \notin \mathfrak{m}$, then the Milnor fiber of f is diffeomorphic to k (disjoint) copies of the Milnor fiber of h. Note that even in this simple case, in general $c_{I,e}(f) \neq 0$. By [23, p. 37] $\tau_{I,e}((h)^k) = J_{h^k} \cap (h)^k$, i.e. $c_{I,e}(h^k) = \mu(h) - \tau(h)$ ($\tau(h)$ = Tjurina number of h), hence $c_{I,e}(h^k) = 0$ if and only if h is weighted homogeneous.

Therefore the $\mathscr{D}_I$-classification is very complicated in general, for example it may happen that we have no $\mathscr{D}_I$-simple germs. (See also [7].)

2.5.4. For germs $g \in \mathfrak{m}$ we have the following

FINITE I-CODIMENSION CRITERION. *Let $f = h^k g \in \int I$ such that $f \in \mathfrak{m}$. Then $c_I(f) < \infty$ if and only if the following two conditions hold:*

(i) *g defines an IS*
(ii) *(h, g) defines an ICIS.*

Proof. Suppose that $c_I(f)$ is finite. Since $\xi(h)/h \in \mathcal{O}$, for $\xi \in TD_I$, we can consider the ideal $\beta = \{kg\xi(h)/h + \xi(g) \mid \xi \in TD_I\}$. From the isomorphism $\int I/\tau_I(f) \approx \mathcal{O}/\beta$ we obtain that β has finite **C**-codimension. Obviously $\beta \subset (g) + J_g$ hence g is an IS. Let c be as in (2.0.2). Then the zero set of the ideal $c + (g) + (h)$ is contained in the zero set of the ideal β, hence (h, g) is an ICIS.

Conversely, if the pair (h, g) is an ICIS and g is an IS then if we denote by J the ideal generated by $\langle kg\partial_i h + h\partial_i g, i = 1, \ldots, n+1 \rangle$ in $\mathcal{O}$, then $J + c$ has finite **C**-codimension. (To see this suppose that $V(J + c) \neq \{0\}$ and use the Curve selection lemma). Considering the derivations $\xi_i = h\partial_i$ and $\xi_{ij} = \partial_i h \cdot \partial_j - \partial_j h \cdot \partial_i$ we obtain that $\beta_e = \{kg\xi(h)/h + \xi(g) \mid \xi \in TD_{I,e}\} \supset c + J$, hence $c_{I,e}(f)$ is finite.

2.5.5. COROLLARY. *If $f = h^k g$ has finite $(h)^{k-1}$-codimension, $g \in \mathfrak{m}$, then the Milnor fiber of f is a bouquet of spheres $S^1 \vee (\bigvee_{\mu_n} S^n)$, where*

$$\mu_n = k(\mu(h, g) + \mu(h)) + \mu(h, g) + \mu(g).$$

Proof. Use (2.5.4) and (2.1.5).

2.5.6. If we consider (h, g) as in (2.1.6), then the pair (z_{n+1}, φ) defines an ICIS, but $c_I(f) = \infty$.

2.5.7. EXAMPLE. Let $h(z) = z_{n+1}$ and $g(z) = \varphi(z_1, \ldots, z_n) + z_{n+1}\psi(z)$. If $\varphi \notin \mathfrak{m}$, then $f \overset{\mathscr{D}_I}{\sim} z_{n+1}^k$; if $\varphi \in \mathfrak{m} - \mathfrak{m}^2$ then $f \overset{\mathscr{D}_I}{\sim} z_{n+1}^k z_n$ (in the latter case the Milnor fiber is homotopic to S^1). In both cases $c_{I,e}(f) = 0$.

If $\varphi \in m^2$, then $c_I(f) < \infty$ if and only if $\varphi\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and $g \in \mathcal{O}$ have only IS and in this case

$$c_{I,e}(f) = c_I(f) - n = \mu(\varphi) + \mu(g) > 0 \qquad ([39]) \tag{2.5.8}$$

2.5.9. EXAMPLE. Let h be a weighted homogeneous germ of degree d_h and weights $w_1, \ldots, w_{n+1}$. Then $TD_{I,e}$ is generated by the Euler derivation $\xi_e = \Sigma_i w_i z_i \partial_i$ and the "trivial derivations" $\xi_{ij} = \partial_i h \cdot \partial_j - \partial_j h \cdot \partial_i$ [5]. If g is also weighted homogeneous with the same weights $w_1, \ldots, w_{n+1}$ and degree d_g, then $\beta_e = c + (g)$, therefore by [20, p. 77] we get again:

$$c_{I,e}(f) = \mu(h, g) + \mu(g).$$

Obviously, when $h \in m^2$, then ξ_e, $\xi_{ij} \in m \operatorname{Der}_{\mathbf{C}}$, hence $c_{I,e}(f) = c_I(f)$.

2.5.10. Let $n=1$, $k=2$ and $h(z_1, z_2) = z_2$. Then our results agree with Siersma's results [29] (for $n=1$), i.e.

$$\mu_1 + 1 = 3\mu(\varphi) + \mu(g) + 1 = \sigma + 2\tau - 1$$

because

$$\sigma = \# A_1 = c_I(f) - 1, \; c_{I,e}(f) = \mu(\varphi) + \mu(g)$$

$$\tau = \# D_\infty = (g, z_2)_0 = \mu(\varphi) + 1.$$

2.5.11. Consider now the case $n=2$ and $k=1$, i.e. $f = gh$. Then $\Sigma_f = V(g) \cap V(h)$ is a 1-dimensional ICIS, hence we can compare our result with [30, 31]. Obviously, the Hessian number $\delta(f)$ vanishes, $\# D_\infty = 0$. By the main theorem of [31], case B, $F \sim S^1 \vee (\bigvee_{\mu_2} S^2)$, where the number of 2-spheres in the bouquet is $\mu(h, g) + \# A_1$ ($\# A_1 = j(f) = \mathbf{dim}_{\mathbf{C}}(h, g)/J_f$). By our result $\# A_1 = \mu(h) + \mu(g) + \mu(h, g)$.

2.5.12. The following (general) statement was formulated as an (erroneous) conjecture in the first version of this paper. Thanks are due to the referee for correcting it and hinting at a way of how to prove it.

If $f = h^k \cdot g$ has finite $(h)^{k-1}$-codimension, then

$$c_{I,e}(f) = \mu(h, g) + \mu(g) + \mu(h) - \tau(h).$$

The proof can be given with a method as in a paper of Greuel and Looijenga: The dimension of smoothing components, *Duke Math. J.* 52 (1985), 263–272.

3. Proofs

3.1. *A TECHNICAL LEMMA.* Let f, h, $g \in \mathscr{L}$, all defined in the ball $\bar{B}_{\varepsilon_0}$. Let $\{0\} \in V \subset V(f)$ be a germ of a real analytic set, also defined in $\bar{B}_{\varepsilon_0}$ such that

$$V(h) \cap V(g) \cap V = \{0\}$$

(in (3.1) we will use the notation $V(h)$ for $\{z \in \bar{B}_{\varepsilon_0} \mid h(z) = 0\}$).

We have the following basic lemma, due essentially to Iomdin [13].

3.1.1. LEMMA. *There exists an $\varepsilon_0 > \varepsilon > 0$ and a neighbourhood G (in B_{ε_0}) of $V - \{0\}$ such that for $z \in B_\varepsilon \cap G - V(f)$ if $\beta_1(h \cdot \overline{\partial h} + g \cdot \overline{\partial g}) + \beta_2 z = \lambda f \cdot \overline{\partial f}$ with $\beta_1, \beta_2 \in \mathbf{R}$, $\beta_1^2 + \beta_2^2 \neq 0$, $\lambda \in \mathbf{C}$ then $\beta_1 \cdot \beta_2 < 0$.*

Proof. The lemma is a consequence of Lemma 1.6. in op. cit. (In the statement of 1.6, V is a complex algebraic set, but this difference is not essential in the proof. See also Lemma 1.5 op. cit. for the real analytic version of the result).

Let $M = \{z \in \bar{B}_\varepsilon \mid$ there exist β_1, $\beta_2 \in \mathbf{R}_+$, $\beta_1 + \beta_2 \neq 0$, and $\lambda \in \mathbf{C}$ such that $\beta_1(h \cdot \overline{\partial h} + g \cdot \overline{\partial g}) + \beta_2 z = \lambda \cdot \overline{\partial f}\}$.

3.1.2. LEMMA. *If ε is sufficiently small then*

$$\overline{M - V(f)} \cap V(f) \subset V(g) \cap V(h)$$

(*the closure is considered in $\bar{B}_\varepsilon$*).

Proof. Suppose the contrary. Then there exists an irreducible component W of $\overline{M - V(f)} \cap V(f)$ such that $W \not\subset V(g) \cap V(h)$ and $d = \mathbf{dim}\, W \cap V(g) \cap V(h) \leqslant \mathbf{dim}\, W - 1$ (all spaces are considered in B_ε with ε sufficiently small). If L is a generic linear space of codimension d which contains the origin, then the real analytic set $V = L \cap W \subset V(f)$ has the property $V \cap V(h) \cap V(g) = \{0\}$ and $\mathbf{dim}\, V \geqslant 1$. But the existence of V is in contradiction with Iomdin's lemma.

3.1.3. REMARK. One of the consequences of (3.1.2) is that the Milnor fibers $f^{-1}(\eta) \cap \bar{B}_\varepsilon$ and $f^{-1}(\eta') \cap \bar{B}_{\varepsilon'}$, $(0 < \varepsilon' < \varepsilon, 0 < \eta \ll \varepsilon, 0 < \eta' \ll \varepsilon')$ are diffeomorphic. For other consequences see [13].

3.2. *Proof of Theorem A*

3.2.1. It is easy to see that it is enough to prove the theorem for $\mu_0 = 1$.

We choose $\varepsilon_1 > 0$ and $\varepsilon_1 \gg \eta_1 > 0$ sufficiently small such that $u = (h,g)\colon (h,g)^{-1}(D_{\eta_1}) \cap \bar{B}_{\varepsilon_1} \to D_{\eta_1}$ is a "good representative" for the ICIS (h,g) ([20]) and in D_{η_1}, $P^{-1}(0)$ and D have cone structure. Let $V(c) \subset B_{\varepsilon_1} \cap u^{-1}(D_\eta)$ be the reduced critical locus of u and $D = u(V(c))$ the corresponding (reduced) discriminant locus.

Let L be a generic line at the origin: $L: \{(c,d)\in D_{\eta_1}: \alpha c+\beta d=0\}$ such that $L\cap D=\{0\}$, $L\cap P^{-1}(0)=\{0\}$. In particular $L(h,g)=\alpha h+\beta g\in \mathcal{O}$ defines an IS.

Let $0<\varepsilon<\varepsilon_1$ be so small that

(3.2.2). (i) B_ε is a Milnor-ball for the functions f and $L(h,g)$.

(ii) $\bar{B}_\varepsilon \subset u^{-1}(D_{\eta_1})$.

(iii) Lemma (3.1.2) is valid for ε for both sets of maps (f,h,g) and $(L(h,g),h,g)$.

Finally we chose $0<\eta<\eta_1$ small enough such that

(3.2.3). (i) $u: u^{-1}(D_\eta)\cap \bar{B}_\varepsilon \to D_\eta$ is a "good representative" for u.

(ii) D_η is a Milnor-ball for $P: (\mathbf{C}^2,0)\to(\mathbf{C},0)$.

(iii) The restriction of $\rho: \mathbf{C}^2\to \mathbf{R}_+$, $\rho((c,d))=|c|^2+|d|^2$ to D has no critical value in $(0,\eta^2]$.

(iv) For all $(c,d)\in D-P^{-1}(0)$, $P^{-1}(P(c,d))$ meets transversely the curve D.

(v) Lemma (2.2.1) is valid for η.

If $\delta>0$ is sufficiently small relative to ε, η_1 and η, then

(i) $F_\delta=f^{-1}(\delta)\cap B_\varepsilon$ is the (open) Milnor fiber of f in B_ε.

(ii) $Q_\delta^1=P^{-1}(\delta)\cap D_{\eta_1}$, $Q_\delta=P^{-1}(\delta)\cap D_\eta$ are the (open) Milnor fibers of P in D_{η_1} resp. D_η.

Obviously $u^{-1}(Q_\delta^1)\cap \bar{B}_\varepsilon=\bar{F}_\delta$.

3.2.4. As a first step we prove that for sufficiently small δ, $u^{-1}(\bar{Q}_\delta)\cap \bar{B}_\varepsilon$ is a deformation retract of $\bar{F}_\delta$. Moreover, $u^{-1}(P^{-1}(\partial\bar{D}_\delta^1)\cap D_\eta)\cap \bar{B}_\varepsilon$ is a deformation retract of $f^{-1}(\partial\bar{D}_\delta^1)\cap \bar{B}_\varepsilon$ by a deformation retract which preserve the fibers of f.

Proof. The difference of $P^{-1}(\partial\bar{D}_\delta^1)\cap D_{\eta_1}-P^{-1}(\partial\bar{D}_\delta^1)\cap D_\eta$ has r connected components, corresponding to the irreducible components of

$$P: C_\delta^i=\{(c,d)\in D_{\eta_1},\ P_i(c,d)\in\partial\bar{D}_\delta^1,\ \eta\leqslant\rho((c,d))\leqslant\eta_1\},\ i=1,\dots,r.$$

For $z\in u^{-1}(C_\delta^i)\cap \bar{B}_\varepsilon$ and δ small, the projection p_z of z and the projection p_ρ of $h\cdot\overline{\partial h}+g\cdot\overline{\partial g}$ on the tangent space $T_z(f^{-1}(f(z)))$ are nonzero by (3.2.2(iii), moreover by this lemma $p_z\notin \mathbf{R}_-\cdot p_\rho$. Therefore we can construct a vector field on $u^{-1}(C_\delta^i)\cap \bar{B}_\varepsilon$ (first locally then by a partition of unity, globally) such that

(a) $v^i(z)$ is tangent to $f^{-1}(f(z))$.

(b) $\mathrm{Re}\langle v^i(z), h\cdot\overline{\partial h}(z)+g\cdot\overline{\partial g}(z)\rangle<0$.

(c) $\mathrm{Re}\langle v^i(z), z\rangle<0$.

For each $z\in u^{-1}(C_\delta^i)\cap \bar{B}_\varepsilon$, there exists a unique smooth path

$$p_z^i: [0,\tau(z)]\to u^{-1}(C_\delta^i)\cap \bar{B}_\varepsilon$$

such that

$$(\mathrm{d}p^i)/(\mathrm{d}t) = v^i(p(t)), \qquad p_z^i(0) = z, \qquad \rho \circ u(p_z^i(\tau(z))) = \eta^2$$

(because $\rho \circ u(p_z^i(t))$ and $\|p_z^i(t)\|$ are increasing functions) and $f(p_z^i(z)) = f(z)$. Then we can define the deformation retract

$$H: [0,1] \times (f^{-1}(\partial \bar{D}_\delta^1) \cap \bar{B}_\varepsilon) \to f^{-1}(\partial \bar{D}_\delta^1) \cap \bar{B}_\varepsilon$$

by

$$H_s(z) = \begin{cases} p_z^i(s \cdot \tau(z)) & \text{if } z \in u^{-1}(C_\delta^i) \cap \bar{B}_\varepsilon \quad i = 1, \ldots, r \\ z & \text{otherwise.} \end{cases}$$

3.2.5. Let us denote the Milnor fiber of $L(h,g)$ by $L_{\delta'} = L(h,g)^{-1}(\delta') \cap B_\varepsilon$. Then similarly as in (3.2.4) we obtain that for δ' sufficiently small $\bar{L}_{\delta'} \cap u^{-1}(\bar{D}_\eta)$ is a deformation retract of $\bar{L}_{\delta'}$.

3.2.6. Let us denote $u^{-1}(\bar{Q}_\delta) \cap \bar{B}_\varepsilon$ by $\bar{\mathscr{F}}_\delta$. Then by construction $u: \bar{\mathscr{F}}_\delta \to \bar{Q}_\delta$ is a smooth fiber bundle over $\bar{Q}_\delta - D$ with fiber the Milnor fiber F_u of the pair (h, g). A singular fiber $u^{-1}((c,d))((c,d) \in \bar{Q} \cap D)$ is also a bouquet of $(n-1)$-spheres and can be obtained from F_u by attaching rank $H_{n-1}(F_u)$ − rank $H_{n-1}(u^{-1}(c,d))$ cells of dimension n.

The cardinal number of the intersection $D \cap \bar{Q}_\delta = \{O_1, \ldots, O_{N_p}\}$ is $N_p = \Sigma_{i=1}^{s-t} (P_i^D, P)_0$ (see (2.0.3) for the notations). It is easy to see that there exists a subspace $\tilde{Q}_\delta = \{$bouquet of circles with base point $*\}$ in $\bar{Q}_\delta$ such that $\tilde{Q}_\delta$ is a deformation retract of $\bar{Q}_\delta$ and $D \cap \tilde{Q}_\delta = \varnothing$. Let D_i, $i = 1, \ldots, N_p$ be small closed C^∞-embedded disks in $\bar{Q}_\delta - \tilde{Q}_\delta$ with center at the points O_i with radius so small that they are mutually disjoint and for all i let l_i be a C^∞-embedded interval in $\bar{Q}_\delta$ from the base point $* \in \tilde{Q}_\delta$ to a point O_i' on ∂D_i such that $l = \bigcup_i l_i$ can be contracted within itself to $*$, $\tilde{Q}_\delta \cup l$ can be contracted within itself to $\tilde{Q}_\delta$ and $\bar{Q}_\delta$ can be contracted to $\tilde{Q}_\delta \cup l \cup \bigcup_i D_i$. Then (by usual arguments, see [28, §7], [31]) we obtain that $\bar{\mathscr{F}}_\delta$ (hence the Milnor fiber $\bar{F}_\delta$, too) has the homotopy type of a space obtained from $u^{-1}(\tilde{Q}_\delta) \cap \bar{B}_\varepsilon$ by attaching cells of dimension n. But $u^{-1}(\tilde{Q}_\delta) \cap \bar{B}_\varepsilon$ may be regarded as the total space of a fiber bundle with fiber $\bar{F}_u$ and base space homotopic to Q_P.

In the following we shall count the number of cells.

3.2.7. Recall that $V(P_i^D)$ $i = 1, \ldots, s-t$ are the irreducible components of $u(\Gamma_f)$ and $\bigcup_{j=1}^{t_i} \Gamma_{ij}$ is the irreducible decomposition of $u^{-1}(V(P_i^D))$. If $(c,d) \in V(P_i^D)$, then $u^{-1}(c,d) \cap \bar{B}_\varepsilon$ is obtained from F_u by attaching $\Sigma\, \mu_{ij}(z)$ cells of dimension n (the sum is taken over $z \in u^{-1}(c,d) \cap \Gamma_f$) [20, pp. 75–76]. Thus each $V(P_i^D)$ contributes with $(P_i^D, P)_0\, \Sigma_{j=1}^{t_i}\, \mu_{ij} d_{ij}$ cells. Therefore the total number of cells is $N = \Sigma_{i=1}^{s-t} (P_i^D, P)_0\, \Sigma_{j=1}^{t_i}\, \mu_{ij} d_{ij}$.

On the other hand

$$(\gamma_f(c), f)_0 = \sum_{i=1}^{s-t} \sum_{j=1}^{t_i} (\gamma_f^{ij}, f)_0 = \sum_{i=1}^{s-t} \sum_{j=1}^{t_i} n(\gamma_f^{ij})((\gamma_f^{ij})_{\text{red}}, f)_0$$
$$= \sum_{i=1}^{s-t} \sum_{j=1}^{t_i} n(\gamma_f^{ij})(P_i^D, P)_0 d_{ij}$$

(see [36, p. 318]), where $n(\gamma_f^{ij})$ is the length of $\mathcal{O}_N$ (N = ideal of nilpotent elements in $\mathcal{O}/\gamma_f^{ij}$).

In order to prove the equality $\mu_{ij} = n(\gamma_f^{ij})$, consider the ICIS

$$(h, g)_z: (\mathbf{C}^{n+1}, z) \to (\mathbf{C}^2, (h, g)(z)),\ z \in \Gamma_{ij} - \{0\}.$$

Then the *reduced* critical locus $\Gamma_{ij,z}^{\text{red}}$ is smooth and $(h, g)_z \mid \Gamma_{ij,z}^{\text{red}}$ is an immersion. Consequently, the rank of $d_z(h, g)$ is not zero, hence at least one component of (h, g) is smooth at z and meets Γ_{ij} transversely at z, hence by [20, 5.11(a)] and using the above formula, we get $\mu_{ij} = n(\gamma_f^{ij})$.

Consequently, the Milnor fiber $\bar{F}_\delta$ has the homotopy type of $u^{-1}(\tilde{Q}_\delta) \cap \bar{B}_\varepsilon$ with $(\gamma_f(c), f)_0$ n-cells attached, which proves (A(a)).

3.2.8. For the definition of the natural map $F \to \bigcup_{\mu_0} K(\pi_1(F), 1)$ see [34, p.427].

3.2.9. Let $n = 1$. Then (A(b)) follows from an Euler-characteristic argument.

3.2.10. Let $n \geqslant 2$ and $\Sigma_f = \emptyset$. Fix δ' as defined in (3.2.5) and take $\delta > 0$ so small that

(a) (3.2.4) is valid.
(b) $\{(c, d) \in \bar{D}_\eta: \alpha c + \beta d = \delta'\} \cap D \subset \{(c, d) \in \bar{D}_\eta: |P(c, d)| \geqslant \delta\}$.
(c) $P: P^{-1}(\partial\bar{D}_\delta^1) \cap \bar{D}_\eta \to \partial\bar{D}_\delta^1$ is a smooth fiber bundle.

Let $\bar{Q}_{|\delta|} = P^{-1}(\partial\bar{D}_\delta^1) \cap \bar{D}_\eta$ and $\bar{Q}_{\geqslant|\delta|} = P^{-1}(\mathbf{C} - D_\delta^1) \cap \bar{D}_\eta$.

By an argument similar to that of [21, pp. 52–53] using moreover (3.2.3) we obtain that the pair $(\bar{Q}_{|\delta|}, \bar{Q}_{|\delta|} \cap D)$ is a deformation retract of $(\bar{Q}_{\geqslant|\delta|}, \bar{Q}_{\geqslant|\delta|} \cap D)$. Lifting this deformation retract we obtain that $\bar{E}_{|\delta|} = u^{-1}(\bar{Q}_{|\delta|}) \cap \bar{B}_\varepsilon$ is a deformation retract of $\bar{E}_{\geqslant|\delta|} = u^{-1}(\bar{Q}_{\geqslant|\delta|}) \cap \bar{B}_\varepsilon$ (see also the argument of [20, 2.D]).

Let $*'$ be a point on $\bar{Q}_\delta \cap \{(c, d): \alpha c + \beta d = \delta'\}$.

As in (3.2.6) we consider the intersection points $\{\alpha c + \beta d = \delta'\} \cap D$, small closed C^∞-embedded, mutually disjoint disks D_i' in $\{\alpha c + \beta d = \delta'\}$ with center at these points and embedded intervals l_i' from $*'$ to points on $\partial D_i'$ such that

(i) $\bigcup_i l_i'$ can be contracted within itself to $*'$.
(ii) $L' = \bigcup_i l_i' \cup \bigcup_i D_i'$ is a deformation retract in $\bar{D}_\eta \cap \{\alpha c + \beta d = \delta'\}$.
(iii) $L' \subset \bar{Q}_{\geqslant|\delta|}$.

Since u is locally trivial over $\bar{D}_\eta \cap \{\alpha c + \beta d = \delta'\} - D$, $u^{-1}(L') \cap \bar{B}_\varepsilon$ has the homotopy type of $\bar{L}_{\delta'} \cap u^{-1}(\bar{D}_\eta)$ which is a deformation retract in $\bar{L}_{\delta'}$ by (3.2.5).

Since $\bar{L}_{\delta'}$ is $(n-1)$-connected (because $L(h, g)$ is IS, see (3.2.1)), the composed morphism, induced by inclusions:

$$\pi_{n-1}(u^{-1}(*') \cap \bar{B}_\varepsilon) \to \pi_{n-1}(u^{-1}(L') \cap \bar{B}_\varepsilon) \to \pi_{n-1}(\bar{E}_{\geqslant|\delta|})$$

is trivial. Obviously, since $\bar{Q}_\delta - D$ is connected, for all $* \in \bar{Q}_\delta - D$ the inclusion $u^{-1}(*) \to \bar{E}_{\geqslant|\delta|}$ induces a trivial morphism at π_{n-1}.

But this is true also for the inclusion $u^{-1}(*) \to \bar{E}_{|\delta|}$ because $\bar{E}_{|\delta|}$ is a deformation retract in $\bar{E}_{\geqslant|\delta|}$.

From the homotopy exact sequence of the Milnor fibration $f^{-1}(\partial\bar{D}^1_\delta) \cap \bar{B}_\varepsilon \xrightarrow{f} \partial\bar{D}^1_\delta$ with fiber $\bar{F}_\delta$, we obtain that

$$\pi_{n-1}(F_\delta) \to \pi_{n-1}(f^{-1}(\partial\bar{D}^1_\delta) \cap \bar{B}_\varepsilon)$$

is injective (recall that $n \geqslant 2$). By (3.2.4) $\bar{E}_{|\delta|}$ is a deformation retract in $f^{-1}(\partial\bar{D}^1_\delta) \cap \bar{B}_\varepsilon$ by a deformation preserving the fibers of f, hence $\pi_{n-1}(\mathscr{F}_\delta) \to \pi_{n-1}(\bar{E}_{|\delta|})$ is also injective. Looking at the composition $u^{-1}(*) \to \bar{\mathscr{F}}_\delta \to \bar{E}_{|\delta|}$ we obtain that $\pi_{n-1}(u^{-1}(*)) \to \pi_{n-1}(\bar{\mathscr{F}}_\delta)$ is also trivial.

3.2.11. Let us summarize our partial results:

—$\bar{\mathscr{F}}_\delta$ has the homotopy type of a space obtained from the total space of the fiber bundle $(T, F_*) = (u^{-1}(\tilde{Q}_\delta) \cap \bar{B}_\varepsilon, F_*) \to (\tilde{Q}_\delta, *)$ by attaching N cells of dimension n such that the morphism (induced by the inclusion) $\pi_{n-1}(F_*) \to \pi_{n-1}(\bar{\mathscr{F}}_\delta)$ is trivial.

—F_* is a bouquet of μ_u spheres of dimension $n-1$ (with natural base point b_0).
—$\tilde{Q}_\delta$ is a bouquet of μ_P circles (with base point $*$).

Let $h_i: F_* \to F_*$ be the geometric monodromies of the fiber bundle corresponding to $i = 1, \ldots, \mu_p$. Then the total space T can be regarded as a factor set of a disjoint union $\bigcup_{i=1}^{\mu_P} \{i\} \times F_* \times [0,1]/\sim$ where $(i, x, 0) \sim (j, x, 0)$, $(i, x, 1) \sim (i, h_i(x), 0)$. We may suppose that $h_i(b_0) = b_0$. Let $\bigvee_{i=1}^{\mu_P} S^1$ be the image of $\bigcup_i \{i\} \times \{b_0\} \times [0,1]$ in T. Then T is obtained from $(\bigvee S^1) \vee F_*$ by attaching $\mu_P \cdot \mu_u$ cells of **dim**.n.

Therefore $\bar{\mathscr{F}}_\delta$ has the homotopy type of a space obtained from $(\bigvee_{i=1}^{\mu_P} S^1) \vee F_*$ by attaching $N + \mu_P \cdot \mu_u$ cells of dimension n, such that $F_* \to \bar{\mathscr{F}}_\delta$ induces the trivial morphism at π_{n-1}. If $n \geqslant 3$, then it is immediate that we may assume that the images of the characteristic maps of all n-cells are in F_*, thus $\bar{\mathscr{F}}_\delta \sim (\bigvee_{\mu_P} S^1) \vee (\bigvee_{\mu_n} S^n)$.

Obviously, the projection $(\bigvee_{\mu_p} S^1) \vee (\bigvee_{\mu_n} S^n) \to \bigvee_{\mu_p} S^1$ can be identified with the map $\bar{\mathscr{F}}_\delta \to \bar{Q}_\delta$.

In order to prove the theorem for $n = 2$, we give the following

3.2.12. DEFINITION. A continuous map $p: E \to B$ between topological Hausdorff spaces is called a m-quasifibration ($0 \leqslant m \leqslant \infty$) if for all points $b \in B$ and $e \in p^{-1}(b)$ the induced maps $p_*: \pi_q(E, p^{-1}(b), e) \to \pi_q(B, b)$ are isomorphisms for $0 \leqslant q \leqslant m-1$ and epimorphisms for $q = m$. (For $m = \infty$ we obtain the usual definition of quasifibration [8]).

A careful inspection of the proof of Satz 2.2 in [8] shows that the following holds true:

3.2.13. CRITERION. Let $\mathscr{U} = (U_i)_{i \in \mathscr{T}}$ be an open covering of B such that

(a) For each $i \in \mathscr{T}$, $p: p^{-1}(U_i) \to U_i$ is a m-quasifibration.
(b) Each non-void intersection $U_i \cap U_j$ can be written as an union of elements in $\mathscr{U}$.

Then p is a m-quasifibration.

3.2.14. Consider the map $\bar{\mathscr{F}}_\delta \to \bar{Q}_\delta$ and an open covering with small disks of the space $\bar{Q}_\delta$: Then $\pi_1(u^{-1}(b)) \to \pi_1(u^{-1}(D))$ is an epimorphism for small disks D, and $b \in D$ hence $\pi_q(u^{-1}(D), u^{-1}(b), e) = 0$ for $q = 0, 1$. Therefore by (3.2.13) u is a 2-quasifibration, in particular $u_*: \pi_1(\bar{\mathscr{F}}_\delta, F_*) \to \pi_1(\bar{Q}_\delta)$ is an isomorphism. But from the following diagram

$$\begin{array}{ccccccc} \pi_1(F_*) & \xrightarrow{\text{trivial}} & \pi_1(\bar{\mathscr{F}}_\delta) & \to & \pi_1(\bar{\mathscr{F}}_\delta, F_*) & \to & 1 \\ & & & & \wr\downarrow u_* & & \\ & & & & \pi_1(\bar{Q}_\delta) & & \end{array}$$

we obtain that $\pi_1(\bar{\mathscr{F}}_\delta) \to \pi_1(\bar{Q}_\delta)$ is an isomorphism.

Consequently, we may assume that the images of the characteristic maps of all 2-cells are in F_*, thus finishing the proof.

3.3. *Proof of Theorem B*

3.3.1. We keep the notations of (3.2). (For simplicity suppose $n \geqslant 2$).

By (2.2.1) we can identify the pairs $(Q_\delta, Q_\delta - D)$ and $(F_\varphi, F_\varphi - D)$ and the representations $\pi_1(Q_\delta - D, *) \to \mathbf{Aut}\, A$ and $\pi_1(F_\varphi - D, *) \to \mathbf{Aut}\, A$ (both induced by ρ).

By (3.2.4) $\bar{E}_{|\delta|}$ is a deformation retract in $f^{-1}(\partial \bar{D}^1_\delta) \cap \bar{B}_\varepsilon$ and (since $P^{-1}(0)$ and D have a cone structure in D_{η_1}) $\bar{E}_{|\delta|} \cap V(c) = f^{-1}(\partial \bar{D}^1_\delta) \cap \bar{B}_\varepsilon \cap V(c)$. We have the following commutative diagram.

$$(3.3.2)\quad \begin{array}{ccc} (\bar{E}_{|\delta|}, \bar{E}_{|\delta|} \cap V(c)) & \xrightarrow[\text{(3.2.4)}]{\text{def. retract}} & (f^{-1}(\partial\bar{D}^1_\delta) \cap \bar{B}_\varepsilon, f^{-1}(\partial\bar{D}^1_\delta) \cap \bar{B}_\varepsilon \cap V(c)) \\ & \searrow^{u} & \\ & (\bar{Q}_{|\delta|}, \bar{Q}_{|\delta|} \cap D) & \\ {\scriptstyle f \mid \bar{E}_{|\delta|}} \searrow & \downarrow P & \swarrow {\scriptstyle f} \\ & \partial\bar{D}^1_\delta & \end{array}$$

Consequently the monodromy of f (over $\partial\bar{D}^1_\delta$) can be identified to the monodromy of $f \mid \bar{E}_{|\delta|}$.

It is easy to see that u is a Thom map over $\partial\bar{D}^1_\delta$. Therefore by the Thom's second isotopy lemma ([41]) u is locally trivial over $\partial\bar{D}^1_\delta$. Hence the geometric monodromy of $f \mid \bar{E}_{|\delta|}$ can be regarded as a lifting of the geometric monodromy of P.

3.3.3. Let T be a small closed tubular neighbourhood of $\bar{Q}_{|\delta|} \cap D$ in $\bar{Q}_{|\delta|}$. Let $u^{-1}(T) \cap \bar{B}_\varepsilon = \mathcal{T}$, $u^{-1}(\partial T) \cap \bar{B}_\varepsilon = \partial\mathcal{T}$, and $\mathcal{C}\mathcal{T} = u^{-1}(\bar{Q}_{|\delta|} - \text{int } T) \cap \bar{B}_\varepsilon$. Then $\bar{E}_{|\delta|} = \mathcal{C}\mathcal{T} \cup \mathcal{T}$ and $\mathcal{C}\mathcal{T} \cap \mathcal{T} = \partial\mathcal{T}$. If we introduce the new strata $\partial\mathcal{T}$ resp. T, then again by the isotopy lemma we may assume that the geometric monodromy of $f \mid \bar{E}_{|\delta|}$ is a lifting of the geometric monodromy of P which preserves the subspaces T, ∂T, $\bar{Q}_{|\delta|} - \text{int } T$. Consequently

$$\zeta_f = \zeta_{\mathcal{T}} \cdot \zeta_{\mathcal{C}\mathcal{T}} \cdot \zeta_{\partial\mathcal{T}}^{-1} \tag{3.3.4}$$

where $\zeta_{\mathcal{T}}$, $\zeta_{\mathcal{C}\mathcal{T}}$ and $\zeta_{\partial\mathcal{T}}$ are the corresponding zeta functions of the monodromy restricted to the spaces $\mathcal{T} \cap \bar{\mathcal{F}}_\delta$, $\mathcal{C}\mathcal{T} \cap \bar{\mathcal{F}}_\delta$ and $\partial\mathcal{T} \cap \bar{\mathcal{F}}_\delta$.

3.3.5. $\mathcal{C}\mathcal{T} \cap \bar{\mathcal{F}}_\delta$ is the total space of a fiber bundle over the base space $\bar{Q}_\delta - \text{int } T$ (which is diffeomorphic to $\bar{Q}_\delta - D$) and with fiber $\bar{F}_u$. By [34, p. 498] there is a convergent E_2 cohomology spectral sequence of bigraded algebras with $E_2^{s,t} = H^s(\bar{Q}_\delta - D, H^t(\bar{F}_u, \mathbf{C}))$ and E_∞ the bigraded algebra associated to some filtration of $H^*(\mathcal{C}\mathcal{T} \cap \bar{\mathcal{F}}_\delta, \mathbf{C})$, but $\bar{Q}_\delta - D$ is a $K(H, 1)$-space, hence $H^s(\bar{Q}_\delta - D, H^t(\bar{F}_u, \mathbf{C})) = H^s(H, H^t(\bar{F}_u, \mathbf{C}))$ which is trivial if $t \neq 0$, $n-1$ or $s \geqslant 2$. (We recall that H^s(free group, M) $= 0$ whenever $s \geqslant 2$ [35, p. 220].) The other groups are:

$$H^q(H, H^0(\bar{F}_u, \mathbf{C})) = H^q(H, \mathbf{C}) = H^q(\bar{Q}_\delta - D, \mathbf{C}) \quad \text{and}$$

$$H^q(H, H^{n-1}(\bar{F}_u, \mathbf{C})) = H^q(H, A) \quad q = 0, 1.$$

Therefore $E_2^{s,t} = E_\infty^{s,t}$ and $\zeta_{\mathscr{C}\mathscr{T}} = \zeta_{P-D} \cdot \zeta_{P,D}$, because the monodromy action on $H^q(H, A)$ $q = 0,1$ can be identified to the action described in diagram (2.2.4). (Here ζ_{P-D} is the zeta function of the monodromy of P induced on $\bar{Q}_\delta - D$.)

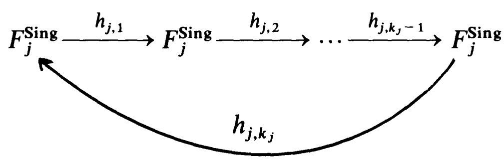
3.3.6. Removing the points $\bar{Q}_\delta \cap D$ in $\bar{Q}_\delta - D$, by the Mayer–Vietoris theorem we obtain that $\zeta_P = \zeta_{P-D} \cdot \zeta_{\bar{Q} \cap T} \cdot (\zeta_{\bar{Q}_\delta \cap \partial T})^{-1}$. Since $\bar{Q}_\delta \cap \partial T$ is a disjoint union of circles, $\zeta_{\bar{Q}_\delta \cap \partial T} = 1$; and $\zeta_{\bar{Q}_\delta \cap T} = \zeta_{\bar{Q}_\delta \cap D}$. But $\bar{Q}_\delta \cap D = \bigcup_{j=1}^{s-t} \{(P, P_i^D)_0 = k_j \text{ points}\}$ and the monodromy can be identified with the cyclic permutation of the points of these sets, therefore by (2.0.5).

$$\zeta_{\bar{Q}_\delta \cap D} = \prod_{j=1}^{s-t} (1 - \lambda^{k_j})^{-1}.$$

Thus

$$\zeta_{\mathscr{C}\mathscr{T}} = \zeta_P \cdot \zeta_{P,D} \cdot \prod_{j=1}^{s-t} (1 - \lambda^{k_j})$$

3.3.7. $u^{-1}(\bar{Q}_\delta \cap D)$ is a deformation retract in $\bar{\mathscr{F}}_\delta \cap \mathscr{T}$, $u^{-1}(\bar{Q}_\delta \cap D) = \bigcup_{j=1}^{s-t} \{k_j$ disjoint copies of $F_j^{\text{Sing}}\}$ and the geometric monodromy is a cyclic map:

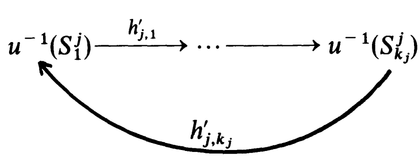


$j = 1, \ldots, s - t$

such that $h_{j,k_j} \circ \cdots \circ h_{j,1}$ induces on $H^{n-1}(F_j^{\text{Sing}}, \mathbf{C})$ the singular monodromy operator h_j^{Sing}. Thus by (2.0.5)

$$\zeta_{\mathscr{T}}(\lambda) = \prod_{j=1}^{s-t} (1 - \lambda^{k_j})^{-1} \prod_{j=1}^{s-t} \mathbf{det}(1 - \lambda^{k_j} h_j^{\text{sing}})^{(-1)^n}$$
$$= \prod_{j=1}^{s-t} (1 - \lambda^{k_j})^{-1} \Delta_j^{\text{Sing}}(\lambda^{k_j}).$$

3.3.8. $\bar{Q}_\delta \cap \partial T = \bigcup_{j=1}^{s-t} \{S_1^j \cup \cdots \cup S_{k_j}^j\}$ where S_i^j are smooth circles around the points of $D \cap \bar{Q}_\delta$. The geometric monodromy on $\bar{\mathscr{F}}_\delta \cap \partial\mathscr{T}$ can be regarded as a cyclic map:



$j = 1, \ldots, s - t$

Therefore $\zeta_{\partial\mathscr{T}}(\lambda) = \Pi_{j=1}^{s-t} \zeta_{h'_{j,k_j}\circ\cdots\circ h'_{j,1}}(\lambda^{k_j})$, but all these zeta functions are trivial due to the following

3.3.9. LEMMA. *Let $E \xrightarrow{P} S^1 \times S^1$ be a fiber bundle. If we consider the application $pr_1 \circ p: E \to S^1$ (pr_1 = first projection) then the zeta function of the monodromy of $\mathrm{pr}_1 \circ p$ is trivial, i.e. $\zeta_{\mathrm{pr}_1\circ p}(\lambda) = 1$.*

Proof. The monodromy induces the following automorphism of the Wang exact sequence of the fibration $E_0 \xrightarrow{\mathrm{pr}_2\circ p} S^1$ with fiber F:

$$\begin{array}{ccccccccc}
\cdots \to & H^q(E_0) & \to & H^q(F) & \xrightarrow{h^*_{1\times S^1} - 1} & H^q(F) & \to & H^{q+1}(E_0) & \to \cdots \\
& h_{\mathrm{pr}_1\circ p}\downarrow & & h^*_{S^1\times 1}\downarrow & & h^*_{S^1\times 1}\downarrow & & h_{\mathrm{pr}_1\circ p}\downarrow & \\
\cdots \to & H^q(E_0) & \to & H^q(F) & \xrightarrow{h^*_{1\times S^1} - 1} & H^q(F) & \to & H^{q+1}(E_0) & \to \cdots
\end{array}$$

where $E_0 = (\mathrm{pr}_1 \circ p)^{-1}(1)$, $F = p^{-1}(1, 1)$.

Thus

$$\zeta_{h\mathrm{pr}_1\circ p} = \zeta_{h^*_{S^1\times 1}} / \zeta_{h^*_{S^1\times 1}} = 1$$

3.3.10. Finally if we summarize:

$$\zeta_f = \zeta_p \cdot \zeta_{P,D} \cdot \prod_{j=1}^{s-t} \Delta_j^{\mathrm{Sing}}(\lambda^{k_j}).$$

3.3.11. If $\Sigma_f = \varnothing$, then $H^q(F, \mathbf{C})$ can be identified to $H^q(Q_\delta, \mathbf{C})$ $q = 0, 1$ by Theorem A(b) hence (b) is trivial.

3.4. *Proof of Theorem C*

3.4.1. Let $b_1, \ldots, b_v$ ($v = \mu(P, D)$) be a system of generators for the group H, and let $g \in G$ such that $\varphi_*(g) = 1$.

If we consider the element g as a "distinguished generator" [10], then we have a presentation of the pair $(G, \langle g \rangle)$:

$$(G, \langle g \rangle) = \langle b_1, \ldots, b_v;\ g\colon g^{-1} b_i^{-1} g c_g(b_i) \quad i = 1, \ldots, v \rangle$$

The Jacobian matrix of $(G, \langle g \rangle)$ is the following (see op. cit.):

$$M = \left| \begin{array}{cccc}
-g^{-1}b_1^{-1} + g^{-1}b_1^{-1}g\dfrac{\partial w_1}{\partial b_1} & g^{-1}b_1^{-1}g\dfrac{\partial w_1}{\partial b_2} & \cdots & g^{-1}b_1^{-1}g\dfrac{\partial w_1}{\partial b_v} \\
g^{-1}b_2^{-1}g\dfrac{\partial w_2}{\partial b_1} & -g^{-1}b_2^{-1} + g^{-1}b_2^{-1}g\dfrac{\partial w_2}{\partial b_2} & \cdots & g^{-1}b_2^{-1}g\dfrac{\partial w_2}{\partial b_v} \\
\vdots & & &
\end{array} \right|$$

If we apply the abelianizer χ, then we obtain the "Jacobian matrix at χ" which is the Alexander matrix of the pair $(G, \langle g \rangle)$:

$$M^{\chi} = \begin{vmatrix} \tilde{\chi}(b_1^{-1}) & & \\ & \chi(b_2^{-1}) & \\ & \vdots & \end{vmatrix} \cdot \begin{vmatrix} -\tilde{\chi}(g^{-1})+\tilde{\chi}\left(\frac{\partial w_1}{\partial b_1}\right) & \tilde{\chi}\left(\frac{\partial w_1}{\partial b_2}\right) & \cdots & \tilde{\chi}\left(\frac{\partial w_1}{\partial b_v}\right) \\ \tilde{\chi}\left(\frac{\partial w_2}{\partial b_1}\right) & -\tilde{\chi}(g^{-1})+\tilde{\chi}\left(\frac{\partial w_2}{\partial b_2}\right) & \cdots & \tilde{\chi}\left(\frac{\partial w_2}{\partial b_v}\right) \\ & \vdots & & \end{vmatrix}$$

If $\chi(b_i)=\Sigma_{j=1}^{r+s-t} a_{ij}e_j$, then $\tilde{\chi}(b_i)=\Pi_j \lambda_j^{a_{ij}}$. Similarly, if $\chi(g)=\Sigma_j a_j e_j$, then $\tilde{\chi}(g)=\Pi_j \lambda_j^{a_j}\cdot$ ($\tilde{\chi}$: $\mathbf{Z}[G]\to\mathbf{Z}[G_{ab}]$ is induced by χ). By [10, 6.3 and 6.4]:

$$\det M^{\chi} = \Delta_*(\lambda_1,\ldots,\lambda_{r+s-t})(1-\tilde{\chi}(g))\left(\bmod \pm \prod_j \lambda_j^{i_j}\right)$$

Since $\tilde{\chi}(b_i^{-1}(i=1,\ldots,v)$ and $\tilde{\chi}(g)$ are invertible elements in $\mathbf{Z}[G_{ab}]$, the determinant of the following matrix

$$\bar{M}^{\chi} = \left[-\delta_{ij} + \bar{\chi}\left(g\frac{\partial w_i}{\partial b_j}\right)\right]_{ij}$$

is also equal to $\Delta_*(\lambda_1,\ldots,\lambda_{r+s-t})(1-\tilde{\chi}(g))$ (mod $\pm\,\Pi_j \lambda_j^{i_j}$).

3.4.2. Let us consider the evaluation map:

$$\varepsilon\colon \mathbf{Z}[G_{ab}] \to \mathbf{End}_{\mathbf{C}} A \otimes_{\mathbf{Z}} \mathbf{Z}[\lambda, \lambda^{-1}]$$

given by

$$\varepsilon(1) = 1 \otimes 1, \qquad \varepsilon(\lambda_i) = \rho_{ab}(e_i) \otimes \lambda^{\varphi_{*,ab}(e_i)} = E_i \otimes \lambda^{m_i}$$

This map extends to

$$\begin{array}{ccc} \mathbf{Z}[G_{ab}] \otimes_{\mathbf{Z}} M_{v\times v}(\mathbf{Z}) & \xrightarrow{\varepsilon\otimes 1} & \mathbf{End}_{\mathbf{C}} A \otimes_{\mathbf{Z}} \mathbf{Z}[\lambda, \lambda^{-1}] \otimes M_{v\times v}(\mathbf{Z}) \\ \| & & \| \\ M_{v\times v}(\mathbf{Z}[G_{ab}]) & & \mathbf{End}_{\mathbf{C}} A \otimes M_{v\times v}(\mathbf{Z}[\lambda, \lambda^{-1}]) \end{array}$$

3.4.3. Let us compute

$$\mathbf{det}(\varepsilon \otimes 1(\bar{M}^{\chi})) = \mathbf{det}\left[-\delta_{ij} + \varepsilon\tilde{\chi}\left(g\frac{\partial w_i}{\partial b_j}\right)\right]_{ij}$$

If $\tilde{\varphi}_*: \mathbf{Z}[G] \to \mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[\lambda, \lambda^{-1}]$ is induced by φ_*, then $\varepsilon \circ \tilde{\chi} = s \circ \tilde{\rho} \otimes \tilde{\varphi}_*$, hence

$$\varepsilon\tilde{\chi}\left(g\frac{\partial w_i}{\partial b_j}\right) = s \circ \tilde{\rho}\left(g\frac{\partial w_i}{\partial b_j}\right) \otimes \tilde{\varphi}_*\left(g\frac{\partial w_i}{\partial b_j}\right).$$

Since

$$\frac{\partial w_i}{\partial b_j} \in \mathbf{Z}[H]\tilde{\varphi}_*\left(\frac{\partial w_i}{\partial b_j}\right) = 1, \text{ hence } \tilde{\varphi}_*\left(g\frac{\partial w_i}{\partial b_j}\right) = \lambda^{\varphi_*(g)} = \lambda.$$

Therefore $\mathbf{det}(\varepsilon \otimes 1(\bar{M}^{\chi})) = \mathbf{det}(-1 + \lambda[g_{\mathrm{Der}}]) = \pm\Delta_{\mathrm{Der}}(\lambda)$.

3.4.4. Since $\varepsilon\,\mathbf{det}\,\bar{M}^{\chi} = \Delta_*(\lambda^m E_1, \ldots, \lambda^{m_{r+s-t}} E_{r+s-t})(1 - \tilde{\rho}(g) \otimes \lambda^{\varphi_*(g)})$ we obtain that $\mathbf{det}(\varepsilon\,\mathbf{det}\,\bar{M}^{\chi}) = \mathbf{det}\,\Delta_* \cdot \Delta_g(\lambda)$.

Hence

$$(\zeta_{P,D}(\lambda))^{(-1)^{n+1}} = \frac{\Delta_{\mathrm{Der}}(\lambda)}{\Delta_g(\lambda)} = \frac{\mathbf{det}(\varepsilon \otimes 1(\bar{M}^{\chi}))}{\mathbf{det}(\varepsilon\,\mathbf{det}\,\bar{M}^{\chi})} \cdot (\pm\mathbf{det}\,\Delta_*)$$

Therefore, Theorem C follows from the following

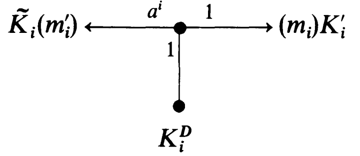
3.4.5. LEMMA. $\mathbf{det}(\varepsilon \otimes 1(M) = \mathbf{det}\,\varepsilon\,\mathbf{det}\,M$ *if* $M \in M_{v \times v}(\mathbf{Z}[G_{ab}])$
Proof. See [4], §12, p. 140, Lemma 2.

3.5. *Proof of Theorem D*

3.5.1. The fiber of P_∞ can be identified to a minimal Seifert surface Q_∞ of the multilink $(S^3, m_1K_1^D \cup \cdots m_tK_t^D \cup m_{t+1}K_{t+1} \cup \cdots \cup m_rK_r)$. Let N_i be a tubular neighbourhood of the link component K_i^D with topological standard meridian M_i and longitude L_i $(i = 1, \ldots, t)$. The minimal Seifert surface Q_∞ meets the torus ∂N_i in d_i parallel copies of $K_i(p_i, q_i)$ where $K_i(p_i, q_i)$ denotes the unique (up to isotopy) oriented simple closed curve in ∂N_i homologous to $p_iL_i + q_iM_i$, $d_i = \text{g.c.d.}\ (m_i, m_i')$, $d_ip_i = m_i$, $d_iq_i = -m_i'$ [9, p. 30].

Let us consider the Seifert links

$\mathcal{S}_i$



Let $\tilde{N}_i$ be a tubular neighbourhood of $\tilde{K}_i$ with topological standard meridian $\tilde{M}_i$ and longitude $\tilde{L}_i$ on $\partial\tilde{N}_i$. Let $\tilde{Q}_i$ be a minimal Seifert surface of the Seifert link

$\mathcal{S}_i$. After splicing, $\tilde{M}_i$ identifies to L_i, $\tilde{L}_i$ to M_i, $\tilde{Q}_i \cap \partial\tilde{N}_i$ to $Q_\infty \cap \partial N_i$ and the fiber $Q_{\mathbf{a}}$ of $P_{\mathbf{a}}$ to the space

$$\left(Q_\infty - \bigcup_{i=1}^{t} \operatorname{int} N_i\right) \cup \bigcup_{\substack{\tilde{Q}_i \cap \partial N_i \\ i=1,\dots,t}} (\tilde{Q}_i - \operatorname{int} \tilde{N}_i)$$

(by the identification of $Q_\infty - \bigcup_{i=1}^t \operatorname{int} N_i$ to $Q_{\mathbf{a}} - \bigcup_{i=1}^t \operatorname{int} N_i$).

3.5.2. The Seifert link $\mathcal{S}_i$ has a realization in the Seifert manifold $\{(z_1, z_2) \in \mathbf{C}^2: |z_1|^2 + |z_2|^2 = 1\} = \Sigma(a^i, 1)$

where

$\tilde{K}_i = \{z_1 = 0\}$, $K_i' = \{z_2 = 0\}$ and K_i^P is a general S^1-orbit, where the S^1-action is given by

$t_*(z_1, z_2) = (tz_1, t^{a^i}z_2)$, $|t| = 1$, $t \in \mathbf{C}$ [op. cit. Ch. 2].

The bord $\partial\tilde{N}_i$ of $\tilde{N}_i$ is given by $|z_1| = \varepsilon$ (for ε small), hence the d_i parallel copies of $d_iK_i(p_i, q_i)$ can be given by the equations:

$$\frac{m_i'}{d_i} \arg z_1 + \frac{m_i}{d_i} \arg z_2 \equiv \frac{2\pi k}{d_i} \pmod{2\pi\mathbf{Z}} \quad k = 0, 1, \dots, d-1.$$

Let $(z_1^0, z_2^0) \in \varphi_i^{-1}(1)$ with $|z_1^0| = \varepsilon$, where

$\varphi_i: \Sigma(a^i, 1) - \{K_i', \tilde{K}_i\} \to S^1$

defined by $\varphi(z_1, z_2) = z_1^{m_i'}z_2^{m_i}/|z_1^{m_i'}z_2^{m_i}|$ is the fibration map of the Seifert manifold. Then the S^1-orbit $(e^{2\pi i\tau}z_1^0, e^{2\pi i a^i\tau}z_2^0)_{\tau\in[0,1]}$ is given by the equations:

$$\arg z_1 \equiv \arg z_1^0 + 2\pi\tau \pmod{2\pi\mathbf{Z}}$$

$$\arg z_2 \equiv \arg z_2^0 + 2\pi a^i\tau \pmod{2\pi\mathbf{Z}}$$

Therefore the S^1-orbit of (z_1^0, z_2^0) meets $d_iK_i(p_i, q_i)$ in $m_i' + a^im_i$ points. The monodromy action $h_i: (z_1, z_2) \to e^{2\pi i/m_i + a^im_i} * (z_1, z_2)$ is a cyclic permutation of these points. Thus, we can denote the points by $T_1^i, \dots, T_{m_i'+a^im_i}^i$ such that $h_i(T_j^i) = T_{j+1}^i$; $j = 1, \dots, m_i' + a^im_i$ (and $T_{m_i'+a^im_i+1}^i = T_1^i$).

3.5.3. The virtual link K_i^P is a general S^1-orbit $t \to t*(z_1^1, z_2^1)$ where $(z_1^1, z_2^1) \in \varphi_i^{-1}(1)$, $|z_1^1| = \varepsilon(1 + r_0)$, $t \in \mathbf{C}$, $|t| = 1$ and $r_0 > 0$ is sufficiently small.

K_i^D meets the fiber $\varphi_i^{-1}(1)$ in $m_i' + a^i m_i$ points given by the equations:

$$z_1 = tz_1^1,\ z_2 = t^{a^i} z_2^1,\ t \in \mathbf{C},\ |t| = 1$$

$$(m_i' + m_i a^i)\,\mathbf{arg}\ t \equiv 0 \pmod{2\pi\mathbf{Z}}$$

These points $\bar{T}_1^i, \ldots, \bar{T}_{m_i'+m_i a^i}^i$ are also circularly permuted by h_i (i.e. $h_i(\bar{T}_j^i) = \bar{T}_{j+1}^i$).

Let $\alpha_i^1 : [0, 1] \to \varphi_i^{-1}(1)$, $i = 1, \ldots, t$ be an embedded segment between T_1^i and $\bar{T}_1^i$ in $\varphi_i^{-1}(1)$ such that $\alpha_i^1(0) = T_1^i$, $\alpha_i^1(1) = \bar{T}_1^i$,

$$|(\alpha_i^1(\tau))_1| \in (\varepsilon, \varepsilon(1 + r_0)) \text{ if } \tau \in (0, 1)$$

Then

$$\alpha_i^j = (h_i)^j \circ \alpha_i^1 : [0, 1] \to \varphi_i^{-1}(1),\ i = 1, \ldots, t;\ j = 1, \ldots, m_i' + a^i m_i,$$

is an embedded segment between T_j^i and $\bar{T}_j^i$.

Finally, let D_{ij} be small closed embedded disks with sufficiently small radius and centers in $\bar{T}_j^i$ such that $R_i = \partial \tilde{N}_i \cap \varphi_i^{-1}(1) \cup \bigcup_j \operatorname{im}\alpha_i^j \cup D_{ij}$ is a deformation retract in $\varphi_i^{-1}(1) - \operatorname{int} \tilde{N}_i$, moreover, the deformation retract may be chosen compatible with the monodromy action.

3.5.4. Since u is locally trivial over $\bigcup_{i=1}^t N_i$ and $Q_\infty - \bigcup_{i=1}^t \operatorname{int} N_i$ is a deformation retract in Q_∞, we obtain that the space $u^{-1}(Q_\infty - \bigcup_{i=1}^t \operatorname{int} N_i)$ is a deformation retract in $u^{-1}(Q_\infty) = F_\infty$.

Since $(Q_\infty - \bigcup_{i=1}^t \operatorname{int} N_i) \cup \bigcup_i R_i$ is a deformation retract in $Q_\mathbf{a}$, and over the difference space u is locally trivial, we obtain that $u^{-1}(Q_\mathbf{a})$ can be replaced by $u^{-1}(Q_\infty) \cup \bigcup_i u^{-1}(\bigcup_j \operatorname{im}\alpha_i^j \cup D_{ij})$ glued over $u^{-1}(\bigcup_j T_j^i)$. Obviously the monodromy preserves this decomposition, hence

$$\zeta_{f_a} = \zeta_{f_\infty} \cdot \prod_{i=1}^{t} \zeta_1^i / \zeta_2^i$$

where

ζ_1^i = zeta function of the monodromy induced on $\bigcup_j u^{-1}(D_{ij})$
ζ_2^i = zeta function of the monodromy induced on $\bigcup_j u^{-1}(T_j^i)$

By Lemma 2.0.5

$$\begin{aligned}\zeta_1^i(\lambda) &= (\text{zeta function induced by the } S^1\text{-orbit } K_i^D \text{ on } u^{-1}(\bar{T}_1^i))(\lambda^{m_i' + m_i a^i}) \\ &= \Delta_i^{\text{Sing}}(\lambda^{m_i' + a^i m_i}) / (1 - \lambda^{m_i' + a^i m_i}).\end{aligned}$$

$\zeta_2^i(\lambda)=$(zeta function induced by the S^1-orbit on

$$u^{-1}(T_1^i))(\lambda^{m_i'+m_ia^i})=\mathbf{det}(1-\lambda^{m_i'+a^im_i}\rho(L_i+a_iM_i))^{(-1)^n}/(1-\lambda^{m_i'+a^im_i})$$

Hence

$$\zeta_{f_a}=\zeta_{f_\infty}\cdot\prod_i[\mathbf{det}(1-\lambda^{m_i'+a^im_i}h_{m_i}^{a^i}h_{m_i})]^{(-1)^{n+1}}\quad\text{(using 2.0.2)}$$

and by (2.0.5) we get formula of Theorem D.

Acknowledgements

I wish to thank A. Zaharia for many helpful conversations related to the present paper. Special thanks are due to Professor D. Eisenbud, who provided me with a copy of his joint book with W. Neumann [9], from which I learned some useful techniques.

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