Szemerédi's Regularity Lemma and its applications in graph theory

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Abstract

Szemerédi's Regularity Lemma is an important tool in discrete mathematics. It says that, in some sense, all graphs can be approximated by random-looking graphs. Therefore the lemma helps in proving theorems for arbitrary graphs whenever the corresponding result is easy for random graphs. Recently quite a few new results were obtained by using the Regularity Lemma, and also some new variants and generalizations appeared. In this survey we describe some typical applications and some generalizations.

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Preface

Szemerédi's Regularity Lemma [121] is one of the most powerful tools of (extremal) graph theory. It was invented as an auxiliary lemma in the proof of the famous conjecture of Erdős and Turán [56] that sequences of integers of positive upper density must always contain long arithmetic progressions. Its basic content could be described by saying that every graph can, in some sense, be well approximated by random graphs. Since random graphs of a given

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edge density are much easier to treat than all graphs of the same edge-density, the Regularity Lemma helps us to carry over results that are trivial for random graphs to the class of all graphs with a given number of edges. It is particularly helpful in "fuzzy" situations, i.e., when the conjectured extremal graphs have no transparent structure.

This paper is partly a survey, partly an attempt to clarify some technical aspects of the Regularity Lemma. It is not aiming at compiling all references on the subject, still we felt that such a pseudo-survey may be useful for graph theorists. We will also provide some proof-sketches to demonstrate how to apply the Regularity Lemma in various situations. We also suggest reading the important paper of Alon, Duke, Leffman, Rödl, and Yuster [3] about the algorithmic aspects of the Regularity Lemma.

Remark. Sometimes the Regularity Lemma is called Uniformity Lemma, see e.g. [64] and [6].

Notation. In this paper we mostly consider simple graphs: graphs without loops and multiple edges.

$v(G)$ is the number of vertices in G (order), $e(G)$ is the number of edges in G (size). G_n will always denote a graph with n vertices. $deg(v)$ is the degree of vertex v and $deg(v, Y)$ is the number of neighbours of v in Y. $\delta(G), \Delta(G)$ and $t(G)$ are the minimum degree, maximum degree and average degree of G. $\chi(G)$ is the chromatic number of G. $N(x)$ is the set of neighbours of the vertex x, and $e(X, Y)$ is the number of edges between X and Y. A bipartite graph G with color-classes A and B and edges E will sometimes be written as $G = (A, B, E)$, $E \subset A \times B$. For disjoint X, Y, we define the **density**

$$d(X,Y) = \frac{e(X,Y)}{|X| \cdot |Y|}\,.$$

The density of a bipartite graph $G = (A, B, E)$ is the number

$$d(G) = d(A,B) = \frac{|E|}{|A| \cdot |B|}\,.$$

$G(U)$ is the restriction of G to U and $G - U$ is the restriction of G to $V(G) - U$. For two disjoint subsets A, B of $V(G)$, we write $G(A, B)$ for the subgraph with vertex set $A \cup B$ which has all edges of G with one endpoint in A and the other in B – when G is clearly understood, we will just call this bipartite graph the pair (A, B). As is customary in graph theory, we will often identify a graph with its edge-set.

For graphs G, H, $H \subset G$ means H is a subgraph of G, but often we will use this in the looser sense that G has a subgraph isomorphic to H (H is embeddable into G), that is, there is a one-to-one map (injection) $\varphi : V(H) \to V(G)$ such that $\{x, y\} \in E(H)$ implies $\{\varphi(x), \varphi(y)\} \in E(G)$. $\|H \to G\|$ denotes the number of labelled copies of H in G. We say that the graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ **pack** (can be packed together) if there is a **bijection** $\varphi : V \to V$ such that $\{x, y\} \in E_1$ implies $\{\varphi(x), \varphi(y)\} \notin E_2$. In other words,

$$G_1 \subset \overline{G_2} := \left(V, \binom{V}{2} - E_2\right).$$

$[n]$ denotes the set $\{1, 2, \ldots, n\}$. The cardinality of a set S will mostly be denoted by $|S|$, but sometimes we write $\#S$. We will be somewhat sloppy by often disregarding rounding.

1 Introduction

1.1 The structure of this survey

Below we start with some historical remarks, then we state and sketch the proof of the Regularity Lemma. After that we introduce the basic notion of the Reduced Graph of a graph corresponding to a partition of the vertex-set, and state a simple but fairly useful tool (Key Lemma). Then in the body of the paper we show how it, or a stronger version of it (Blow-up Lemma), can be used for building bounded degree subgraphs H in a large dense graph G_n, as well as for embedding trees. This will provide simple proofs for many classical and new theorems of extremal graph theory.

We will also touch upon some algorithmic aspects of the Regularity Lemma, its relation to quasi-random graphs and extremal subgraphs of a random graph. We also shortly mention a sparse version and a hypergraph version.

1.2 Regular pairs

Regular pairs are highly uniform bipartite graphs, namely ones in which the density of any reasonably sized subgraph is about the same as the overall density of the graph.

Definition 1.1 (Regularity condition). *Let $\varepsilon > 0$. Given a graph G and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair (A, B) is ε-regular if for every $X \subset A$ and $Y \subset B$ satisfying*

$$|X| > \varepsilon|A| \quad \text{and} \quad |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

The following simple fact guarantees that it is sufficient to check the regularity condition for sets of exact size $|X| = \lfloor \varepsilon|A| \rfloor + 1$, $|Y| = \lfloor \varepsilon|B| \rfloor + 1$.

Fact 1.2 (Convexity of density). *Given a bipartite graph with colour classes A and B, for all integers $k < |A|$ and $\ell < |B|$,*

$$d(A, B) = \frac{1}{\binom{|A|}{k}\binom{|B|}{\ell}} \sum (d(X, Y) : X \subset A, |X| = k, Y \subset B, |Y| = \ell).$$

The next one is the most important property of regular pairs.

Fact 1.3. *(Most degrees into a large set are large) Let (A, B) be an ε-regular pair with density d. Then for any $Y \subset B$, $|Y| > \varepsilon|B|$ we have*

$$\#\{x \in A : deg(x, Y) \leq (d - \varepsilon)|Y|\} \leq \varepsilon|A|.$$

More generally, if we fix a $Y \subset B$, and ℓ vertices $x_i \in A$, then "typically" they have at least the expected $d^\ell|Y|$ neighbours in common.

Fact 1.4 (Intersection Property). *If $Y \subset B$ and $(d - \varepsilon)^{\ell-1}|Y| > \varepsilon|B|$, $(\ell \geq 1)$, then*

$$\#\left\{(x_1, x_2, \ldots, x_\ell) : x_i \in A, \left|Y \bigcap \left(\bigcap_{i=1}^{\ell} N(x_i)\right)\right| \leq (d - \varepsilon)^\ell|Y|\right\} \leq \ell\varepsilon|A|^\ell.$$

The last two properties have corresponding upper parts (e.g. $deg(x, Y) \leq (d - \varepsilon)|Y|$ replaced by $deg(x, Y) \geq (d + \varepsilon)|Y|$), but we usually use them the way we stated them, and in these forms they also hold for somewhat weaker structures.

The next property says that subgraphs of a regular pair are regular.

Fact 1.5 (Slicing Lemma). *Let (A, B) be an ε-regular pair with density d, and, for some $\alpha > \varepsilon$, let $A' \subset A$, $|A'| \geq \alpha|A|$, $B' \subset B$, $|B'| \geq \alpha|B|$. Then (A', B') is an ε'-regular pair with $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$, and for its density d' we have $|d' - d| < \varepsilon$.*

Later we will also use a one-sided version of regularity:

Definition 1.6 (Super-regularity). *Given a graph G and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair (A, B) is (ε, δ)-super-regular if for every $X \subset A$ and $Y \subset B$ satisfying*

$$|X| > \varepsilon|A| \quad \text{and} \quad |Y| > \varepsilon|B|$$

we have

$$e(X, Y) > \delta|X||Y|,$$

and furthermore,

$$deg(a) > \delta|B| \quad \text{for all} \quad a \in A, \quad \text{and} \quad deg(b) > \delta|A| \quad \text{for all} \quad b \in B.$$

1.3 The Regularity Lemma

The Regularity Lemma says that every dense graph can be partitioned into a small number of regular pairs and a few leftover edges. Since regular pairs behave as random bipartite graphs in many ways, the Regularity Lemma provides us with an approximation of an arbitrary dense graph with the union of a constant number of random-looking bipartite graphs.

Theorem 1.7 (Regularity Lemma, Szemerédi 1978 [121]). *For every $\varepsilon > 0$ and m there exist two integers $M(\varepsilon, m)$ and $N(\varepsilon, m)$ with the following property: for every graph G with $n \geq N(\varepsilon, m)$ vertices there is a partition of the vertex set into $k + 1$ classes*

$$V = V_0 + V_1 + V_2 + \ldots + V_k$$

such that

- $m \leq k \leq M(\varepsilon, m)$,
- $|V_0| < \varepsilon n$,
- $|V_1| = |V_2| = \ldots = |V_k|$,
- *all but at most* εk^2, *of the pairs* (V_i, V_j) *are* ε*-regular.*

The classes V_i will be called **groups** or **clusters**. The role of the exceptional set V_0 is purely technical: to make possible that all other classes have *exactly* the same cardinality. Indeed, having an m and choosing $m' > m, \varepsilon^{-2}$ and applying the Regularity Lemma with this new m, one can distribute the vertices of V_0 evenly among the other classes so that $|V_i| \approx |V_j|$ and ε-regularity is preserved with a slightly larger ε. In other words, we may assume that $V_0 = \emptyset$ if the conditions $|V_i| = |V_j|$ are relaxed to $||V_i| - |V_j|| \leq 1$.

The role of m is to make the classes V_i sufficiently small, so that the number of edges inside those classes are negligible. Hence, the following is an alternative form of the Regularity Lemma.

Theorem 1.8. (*Regularity Lemma – alternative form*) *For every* $\varepsilon > 0$ *there exists an* $M(\varepsilon)$ *such that the vertex set of any* n*-graph* G *can be partitioned into* k *sets* $V_1, \ldots, V_k$, *for some* $k \leq M(\varepsilon)$, *so that*

- $|V_i| \leq \lceil \varepsilon n \rceil$ *for every* i,
- $||V_i| - |V_j|| \leq 1$ *for all* i, j,
- (V_i, V_j) *is* ε*-regular in* G *for all but at most* εk^2 *pairs* (i, j).

If we have a sequence (G_n) of graphs with $e(G_n) = o(n^2)$, the Regularity Lemma becomes trivial: G_n is approximated by the empty graph. Thus the Regularity Lemma is useful only for **large, dense graphs.**

Definition 1.9 ([116]). *Given an* $r \times r$ *symmetric matrix* (p_{ij}) *with* $0 \leq p_{ij} \leq 1$, *and positive integers* $n_1, \ldots, n_r$, *we define a* **generalized random graph** R_n (*for* $n = n_1 + \ldots + n_r$) *by partitioning* n *vertices into classes* V_i *of size* n_i *and then joining the vertices* $x \in V_i$, $y \in V_j$ *with probability* p_{ij}, *independently for all pairs* $\{x, y\}$.

Now, Szemerédi's Lemma asserts in a way that every graph can be approximated by generalized random graphs.

1.4 A more applicable form of the Regularity Lemma

Most applications of the Regularity Lemma deal with monotone problems, when throwing in more edges can only help. In these applications, one starts with applying the original form of the Regularity Lemma to create a regular partition, then gets rid of all edges within the clusters of the partition, also the edges of non-regular pairs as well as those of regular pairs with too low densities. The leftover "pure" graph is much easier to handle and it still contains most of the original edges. The following precise formulation of this process is a simple consequence of the Regularity Lemma

Theorem 1.10 (Degree Form). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex-set V into $k + 1$ clusters $V_0, V_1, \dots, V_k$, and there is a subgraph $G' \subset G$ with the following properties:*

- $k \le M$,
- $|V_0| \le \varepsilon|V|$,
- *all clusters* V_i, $i \ge 1$, *are of the same size* $m \le \lceil \varepsilon|V| \rceil$,
- $deg_{G'}(v) > deg_G(v) - (d + \varepsilon)|V|$ *for all* $v \in V$,
- $e(G'(V_i)) = 0$ *for all* $i \ge 1$,
- **all** *pairs* $G'(V_i, V_j)$ $(1 \le i < j \le k)$ *are ε-regular, each with a density either 0 or greater than* d.

Remark. In a typical application of the Degree Form, we start off with a graph G_n and appropriate parameters ε and d, and then obtain a partition $V_0, V_1, \dots, V_k$ of V. Then we usually drop the set V_0 to get a **"pure"** graph $G'' = G' - V_0$. This pure graph G'' is much easier to deal with, and it still contains most of the original edges:

$$deg_{G''}(v) > deg_G(v) - (d + \varepsilon)n - |V_0| \ge deg_G(v) - (d + 2\varepsilon)n \quad \text{for all} \quad v \in V(G''),$$

whence

$$e(G'') > e(G) - (d + 3\varepsilon)n^2/2.$$

1.5 The road to the Regularity Lemma

The following is a basic result in combinatorial number theory.

Theorem 1.11 (van der Waerden 1927 [125]). *Let k and t be arbitrary positive integers. If we color the integers in t colors, at least one color-class will contain an arithmetic progression of k terms.*

A standard compactness argument shows that the following is an equivalent form.

Theorem 1.12 (van der Waerden - finite version). *For any integers k and t there exists an n such that if we color the integers $\{1, \dots, n\}$ with t colors, then at least one color-class will contain an arithmetic progression of k terms.*

This is a Ramsey type theorem in that it only claims the existence of a given configuration in one of the color classes without getting any control over which class it is. It turns out that the van der Waerden problem is *not* a true Ramsey type question but of a density type: the only thing that matters is that at least one of the color classes contains relatively many elements. Indeed, answering a very deep and difficult conjecture of P. Erdős and P. Turán from 1936 [56], Endre Szemerédi proved that positive upper density implies the existence of an arithmetic progression of k terms.

Theorem 1.13 (Szemerédi 1975 [120]). *For every integer $k > 2$ and $\varepsilon > 0$ there exists a threshold $n_0 = n_0(k, \varepsilon)$ such that if, for some $n \geq n_0$, $A \subset \{1, \ldots, n\}$ and $|A| > \varepsilon n$, then A must contain an arithmetic progression of k terms.*

Remark. For $k = 3$ this is a theorem of K.F. Roth [101] that dates back to 1954, and it was already an important breakthrough when Szemerédi succeeded in proving the theorem in 1969 for $k = 4$ [118]. One of the interesting questions in this field is the speed of convergence to 0 of $r_k(n)/n$, where $r_k(n)$ is the maximum size of a subset of $[n]$ not containing an arithmetic progression of length k. Szemerédi's proof used van der Waerden's theorem and therefore gave no reasonable bound on the convergence rate of $r_4(n)/n$. Roth found an analytical proof a little later [102, 103] not using van der Waerden's theorem and thus providing some weak estimates on the convergence rate of $r_4(n)/n$ [102], which - probably - imply that

$$r_4(n) = O\left(\frac{n}{\log_\ell n}\right)$$

for some sufficiently large ℓ, where $\log_\ell n$ denotes the ℓ-times iterated logarithm.

Szemerédi's theorem was also proved by Fürstenberg [67] in 1977 using ergodic theoretical methods. It was not quite clear first if the Fürstenberg proof was really different from that of Szemerédi, but subsequent generalizations due to Fürstenberg and Katznelson [69] and later by Bergelson and Leibman [10] convinced the mathematical community that Ergodic Theory is a natural tool to attack combinatorial questions. The scope of this survey does not allow us to explain these generalizations. We refer the reader to the book of R.L. Graham, B. Rothschild and J. Spencer, *Ramsey Theory* [71], which describes the Hales-Jewett theorem and how these theorems are related, and its chapter "Beyond Combinatorics" gives an introduction into related fields of topology and ergodic theory. Another good source is the paper of Fürstenberg in this very volume [68].

1.6 A historical detour: the Original Szemerédi Lemma

To prove his theorem $r_k(n) = o(n)$, Szemerédi used a weaker version of his lemma [120], which was formulated only for bipartite graphs.

Theorem 1.14 (The Old Szemerédi Lemma). *For every $\varepsilon_1, \varepsilon_2, \delta, \varrho, \sigma > 0$ there exist n_0, m_0, N, M, such that for every bipartite graph (A, B, E) with $|A| = n \geq N$, $|B| = m \geq M$ there exist sets $V_i \subset A$ $(i < n_0)$ and $V_{ij} \subset B$ $(i < n_0,\ j < m_0)$ for which*

(a) $|A - \cup_{i<n_0} V_i| < \varrho n$, *and* $|B - \bigcup_{j<m_0} V_{ij}| < \sigma m$, *for every* $i < n_0$, *and*
(b) *for every* $i < n_0$, $j < m_0$ *and for every* $T \subset V_i$, $S \subset V_{ij}$, *if* $|T| > \varepsilon_1 |V_i|$ *and* $|S| > \varepsilon_2 |V_{ij}|$, *then*

$$d(T, S) > d(V_i, V_{ij}) - \delta$$

and

$$|N(u) \cap V_i| < (d(V_i, V_{ij}) + \delta)|V_i| \quad \text{for each} \quad u \in V_{ij}.$$

The Old Szemerédi Lemma is weaker than the new version not because it refers to bipartite graphs but because for each V_i of the partition we have to choose its own partition V_{ij} of the set $V(G_n) - V_i$. Still there are many cases where the Old Szemerédi Lemma is as applicable as the new one; see [119, 108].

In this paper we consider almost exclusively graph theoretical applications. However, the lemma was invented to solve number-theoretical problems, and it is still used for this purpose also. For a recent number theoretical application see the paper of Balog and Szemerédi [9]. There are also many applications in combinatorial geometry. We refer the reader to a forthcoming book of Pach and Agarwal [96], and to the paper of Erdős, Makai and Pach [50].

1.7 Proof of the Regularity Lemma

We only sketch the proof and emphasize its main features.

First a measure called **index** is defined for every partition of $V(G)$ measuring in a way how regular the pairs are in the partition. Let P be a partition of V into $V_0, V_1, \ldots, V_k$ and let

$$\operatorname{ind}(P) = \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=i+1}^{k} d^2(V_i, V_j).$$

Obviously,

$$\operatorname{ind}(P) \leq \frac{1}{2}.$$

Now the basic idea is that if a partition violates the regularity conditions of the Regularity Lemma then one can refine this partition so that the index will grow significantly.

Lemma 1.15. *Let $G = (V, E)$ be a graph with n vertices. Let P be a partition of V into $k+1$ classes $V_0, V_1, \ldots, V_k$ (where $k \geq k_0$) so that $|V_0| < \varepsilon n$ and the V_i's have the same size for $1 \leq i \leq k$. If for a given $\varepsilon > 0$ more than εk^2 classes are ε-irregular* [1]*, then there exists a refinement Q of P* [2] *into $1 + k4^k$ classes such that*

$$\operatorname{ind}(Q) \geq \operatorname{ind}(P) + \frac{\varepsilon^5}{20} \tag{1}$$

and the size of the exceptional class V_0 increases by at most $n/4^k$.

[1] i.e., not ε-regular
[2] More precisely, Q is a refinement if we disregard the exceptional class V_0

Iterating this refinement in t steps and using (1) we get for the t-th new partition P_t:

$$\frac{1}{2} \geq \operatorname{ind}(P_t) \geq \operatorname{ind}(P) + \frac{t\varepsilon^5}{20}$$

implying

$$t \leq \frac{10}{\varepsilon^5}.$$

Hence in at most $10\varepsilon^{-5}$ improvement steps we arrive at a partition which satisfies the conditions of the lemma. This means that the number of classes (disregarding the exceptional class V_0) will be "t-times iterated exponentiation": Define $f(0) = m$, $f(t+1) = 1 + f(t) \cdot 4^{f(t)}$. Then the number of classes will be at most $f(10/\varepsilon^5)$.

The proof of Lemma 1.15 uses the following **defect** form of the Cauchy-Schwarz inequality.

Lemma 1.16 (Improved Cauchy-Schwarz inequality). *If for the integers* $0 < m < n$,

$$\sum_{k=1}^{m} X_k = \frac{m}{n} \sum_{k=1}^{n} X_k + \delta,$$

then

$$\sum_{k=1}^{n} X_k^2 \geq \frac{1}{n} \left(\sum_{k=1}^{n} X_k \right)^2 + \frac{\delta^2 n}{m(n-m)}.$$

The following simple property of density is also used.

Fact 1.17 (Continuity of density).

$$|d(X,Y) - d(A,B)| \leq (1 - |X|/|A|) + (1 - |Y|/|B|) \quad \textit{for all} \quad X \subset A,\ Y \subset B.$$

One remark should be made here. As we have seen, the proof of the Regularity Lemma involved $10\varepsilon^{-5}$-times iterated exponentiation. Hence the estimates in many applications of Regularity Lemma seem to be too weak. But in the original application, namely in the proof of $r_k(n) = o(n)$, this is not the weakest point: Szemerédi applied the van der Waerden Theorem where the estimates are much-much weaker. For a much weaker but quantitatively more efficient statement, see Lemma 7.5.

1.8 Are there exceptional pairs ?

The Regularity Lemma does not assert that all pairs of clusters are regular. In fact, it allows εk^2 pairs to be irregular. For a long time it was not known if there must be irregular pairs at all. It turned out that there must be at least ck irregular pairs. Alon, Duke, Leffman, Rödl and Yuster [3] write: "In [121] the author raises the question if the assertion of the lemma holds when we do not allow any irregular pairs in the definition of a regular partition. This, however, is not true, as observed by several researchers, including L. Lovász, P. Seymour, T. Trotter and ourselves. A simple example showing that irregular pairs are necessary is a bipartite graph with vertex classes $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ in which $a_i b_j$ is an edge iff $i \leq j$." [3]

[3] This important graph is called the **half-graph.**

1.9 The Regularity Lemma with many colors

Some generalizations use an extension of the Regularity Lemma for many colors. This asserts that if the edges are r-colored, then we may partition the vertex-set into a bounded number of classes so that almost all pairs of classes are ε-regular in each color simultaneously. If the edges of a graph are r-colored, we will write d_ν for the edge-density in the ν-th color.

Theorem 1.18 (Many-Color Regularity Lemma). *For any $\varepsilon > 0$ and integers r, κ there exists an M such that if the edges of a graph G_n are r-colored then the vertex set $V(G_n)$ can be partitioned into sets $V_0, V_1, \ldots, V_k$ – for some $\kappa \le k \le M$ – so that $|V_0| < \varepsilon n$, $|V_i| = m$ (the same) for every $i \ge 1$, and all but at most εk^2 pairs (V_i, V_j) satisfy the following regularity condition: for every $X \subset V_i$ and $Y \subset V_j$ of size $|X|, |Y| > \varepsilon m$ we have*

$$|d_\nu(X, Y) - d_\nu(V_i, V_j)| < \varepsilon \qquad (\nu = 1, \ldots, r).$$

Proof. Use the original proof, but modify the definition of index by summing the indices for each color: for a partition P of V into $V_0, V_1, \ldots, V_k$, let

$$\mathrm{ind}(P) = \frac{1}{k^2} \sum_\nu \sum_{i=1}^{k} \sum_{j=i+1}^{k} d_\nu^2(V_i, V_j).$$ ∎

2 How to apply the Regularity Lemma

2.1 The Reduced Graph

Given an arbitrary graph $G = (V, E)$, a partition P of the vertex-set V into $V_1, \ldots, V_k$, and two parameters ε, d, we define the **Reduced Graph** (or Cluster graph) R as follows: its vertices are the clusters $V_1, \ldots, V_k$ and V_i is joined to V_j if (V_i, V_j) is ε-regular with density more than d. Most applications of the Regularity Lemma use Reduced Graphs, and they depend upon the fact that many properties of R are inherited by G. Typically, we start with "purifying the graph" as described after Theorem 1.10, that is, we pass from the graph G to G'' (or G') of the Degree Form, and study the Reduced Graph of that latter graph. The most important property of Reduced Graphs is mentioned in the following section.

2.2 A useful lemma

Many of the proofs using the Regularity Lemma struggle through similar technical details. These details are often variants of an essential feature of the Regularity Lemma: If G has a reduced graph R and if the parameter ε is small enough, then every small subgraph H of R is also a subgraph of G. In the first applications of the Regularity Lemma the graph H was fixed, but the greedy algorithm outlined in the section "Building up small subgraphs" works smoothly even when the order of H is proportional with that of G as long as H has bounded degrees. (Another standard class of applications – embedding trees into dense graphs – will be discussed later.)

The above mentioned greedy embedding method for bounded degree graphs is so frequently used that, just to avoid repetitions of technical details, it is worth while spelling it out in a quotable form.

For a graph R and positive integer t, let $R(t)$ be the graph obtained from R by replacing each vertex $x \in V(R)$ by a set V_x of t independent vertices, and joining $u \in V_x$ to $v \in V_y$ iff (x, y) is an edge of R. In other words, we replace the edges of R by copies of the complete bipartite graph K_{tt}.

Theorem 2.1 (Key Lemma). *Given $d > \varepsilon > 0$, a graph R, and a positive integer m, let us construct a graph G by replacing every vertex of R by m vertices, and replacing the edges of R with ε-regular pairs of density at least d. Let H be a subgraph of $R(t)$ with h vertices and maximum degree $\Delta > 0$, and let $\delta = d - \varepsilon$ and $\varepsilon_0 = \delta^\Delta/(2 + \Delta)$. If $\varepsilon \le \varepsilon_0$ and $t - 1 \le \varepsilon_0 m$, then $H \subset G$. In fact,*

$$\|H \to G\| > (\varepsilon_0 m)^h.$$

Remark. Note that $v(R)$ didn't play any role here.

Remark. Often we use this for R itself (that is, for $t = 1$): If $\varepsilon \le \delta^{\Delta(R)}/(2 + \Delta(R))$ then $R \subset G$, in fact, $\|R \to G\| \ge (\varepsilon m)^{v(R)}$.

Remark. Using the Slicing Lemma (and changing the value of ε_0), it is easy to replace the condition $H \subset R(\varepsilon_0 m)$ with the assumptions

(*) every component of H is smaller than $\varepsilon_0 m$,

(**) $H \subset R((1 - \varepsilon_0)m)$.

One can strengthen this tremendously by proving the same for all bounded degree subgraphs H of the full $R(m)$. This provides a very powerful tool (Blow-up Lemma), and it is described in Section 6.

Proof of the Key Lemma. We prove the following more general estimate.

$$\text{If} \quad t - 1 \le (\delta^\Delta - \Delta\varepsilon)m \quad \text{then} \quad \|H \to G\| > \left[(\delta^\Delta - \Delta\varepsilon)m - (t-1)\right]^h.$$

We embed the vertices $v_1, \ldots, v_h$ of H into G by picking them one-by-one. For each v_j not picked yet we keep track of an ever shrinking set C_{ij} that v_j is confined to, and we only make a final choice for the location of v_j at time j. At time 0, C_{0j} is the full m-set v_j is *a priori* restricted to in the natural way. Hence $|C_{0j}| = m$ for all j. The algorithm at time $i \ge 1$ consists of two steps.

Step 1 - *Picking v_i.* We pick a vertex $v_i \in C_{i-1,i}$ such that

$$deg_G(v_i, C_{i-1,j}) > \delta|C_{i-1,j}| \quad \text{for all} \quad j > i \quad \text{such that} \quad \{v_i, v_j\} \in E(H). \tag{1}$$

Step 2. - *Updating the C_j's.* We set, for each $j > i$,

$$C_{ij} = \begin{cases} C_{i-1,j} \cap N(v_i) & \text{if } \{v_i, v_j\} \in E(H) \\ C_{i-1,j} & \text{otherwise.} \end{cases}$$

For $i < j$, let $d_{ij} = \#\{\ell \in [i] : \{v_\ell, v_j\} \in E(H)\}$.

Fact. *If $d_{ij} > 0$ then $|C_{ij}| > \delta^{d_{ij}} m$.* (If $d_{ij} = 0$ then $|C_{ij}| = m$.)

Thus, for all $i < j$, $|C_{ij}| > \delta^{\Delta} m \geq \varepsilon m$, and hence, when choosing the exact location of v_i, all but at most $\Delta \varepsilon m$ vertices of $C_{i-1,i}$ satisfy (1). Consequently, we have at least

$$|C_{i-1,i}| - \Delta\varepsilon m - (t-1) > (\delta^{\Delta} - \Delta\varepsilon)m - (t-1)$$

free choices for v_i, proving the claim. ∎

Remark. We did not use the full strength of ε-regularity for the pairs (A, B) of m-sets replacing the edges of H, only the following one-sided property:

$$X \subset A, |X| > \varepsilon|A|,\ Y \subset B, |Y| > \varepsilon|B| \quad \text{imply} \quad e(X, Y) > \delta|X||Y|.$$

Now most applications start with applying the Regularity Lemma for a graph G and finding the corresponding Reduced Graph R. Then usually a classical extremal graph theorem (like the König-Hall theorem, Dirac's theorem, Turán's theorem or the Hajnal-Szemerédi theorem) is applied to the graph R. Then an argument similar to the Key Lemma (or its strengthened version, the Blow-up Lemma) is used to lift the theorem back to the graph G.

2.3 Some classical extremal graph theorems

This is only a brief overview of the standard results from extremal graph theory most often used in applications of the Regularity Lemma. For a detailed description of the field we refer the reader to [11, 115, 65].

The field of extremal graph theory started with the historical paper of Pál Turán in 1941, in which he determined the minimal number of edges that guarantees the existence of a p-clique in a graph. The following form is somewhat weaker than the original theorem of Turán, but it is perhaps the most usable form.

Theorem 2.2 (Turán 1941 [124]). *If G_n is a graph with n vertices and*

$$e(G) > \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2},$$

then $K_p \subset G_n$.

In general, given a family $\mathcal{L}$ of **excluded** graphs, one would like to determine the maximum number of edges a graph G_n can have without containing any subgraph $L \in \mathcal{L}$. This maximum is denoted by $\text{ex}(n, \mathcal{L})$ and the graphs attaining the maximum are called **extremal graphs**. (We will use the notation $\text{ex}(n, \mathcal{L})$ for hypergraphs, too.) These problems are often called Turán type problems, and are mostly considered for **simple graphs or hypergraphs**, but there are also many results for **multigraphs and digraphs** of bounded edge- or arc-multiplicity (see e.g. [18, 19, 20, 21, 112]).

Using this notation, the above form of Turán's theorem says that

$$\text{ex}(n, K_p) \le \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}.$$

The following theorem of Erdős and Stone determines $\text{ex}(n, K_p(t, \ldots, t))$ asymptotically.

Theorem 2.3 (Erdős-Stone 1946 [55] – Weak Form). *For any integers $p \ge 2$ and $t \ge 1$,*

$$\text{ex}(n, K_p(t, \ldots, t)) = \left(1 - \frac{1}{p-1}\right)\binom{n}{2} + o(n^2).$$

(For strengthened versions, see [28, 29].) This is, however, much more than just another Turán type extremal result. As Erdős and Simonovits pointed out in [52], it implies the general asymptotic description of $\text{ex}(n, \mathcal{L})$.

Theorem 2.4. *If $\mathcal{L}$ is finite and $\min_{L \in \mathcal{L}} \chi(L) = p > 1$, then*

$$\text{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p-1}\right)\binom{n}{2} + o(n^2).$$

So this theorem plays a crucial role in extremal graph theory. (For structural generalizations for arbitrary $\mathcal{L}$ see [36, 37, 113].) Its basic message is that the critical parameter determining whether a graph L is a subgraph of *all* graphs G_n with a given edge density, is the chromatic number of L. Contrast this with the well-known fact that the corresponding parameter for *random* G_n is the average degree of L (more precisely, the maximum of the average degrees of all subgraphs of L).

The following is a generalization of the Erdős-Stone theorem for hypergraphs, where $L(t)$ is the hypergraph obtained from L as described for graphs in the paragraph preceding the Key Lemma.

Theorem 2.5 (Erdős [38], Brown-Simonovits [21]). *For r-uniform hypergraphs*

$$\text{ex}(n, L(t)) = \text{ex}(n, L) + o(n^r).$$

Another classical theorem often applied in proofs employing the Regularity Lemma is Dirac's theorem.

Theorem 2.6 (Dirac 1952 [32]). *If an n-graph G has minimum degree at least $n/2$ then G is Hamiltonian.*

Just as Turán's theorem or Dirac's theorem are the standard tools in simple applications of the Regularity Lemma, the following deep theorem of Hajnal and Szemerédi is the key element in sophisticated applications.

Theorem 2.7 (Hajnal-Szemerédi 1969 [73], – Complementary Form). *If $\delta(G_n) \geq (1-1/r)n$ then G_n contains $\lfloor n/r \rfloor$ vertex-disjoint copies of K_r.*

2.4 Two short proofs

While the Erdős-Stone theorem has numerous classical proofs, just for demonstrating the standard Regularity Lemma argument let us show how the Erdős-Stone theorem follows quite easily from Turán's theorem and the Regularity Lemma.

Let $\beta > 0$, and let G_n have more than $(1-\frac{1}{p-1}+\beta)\binom{n}{2}$ edges, where n is large enough. Apply the Degree Form of the Regularity Lemma (Theorem 1.10) with $d = \beta/2$ and $\varepsilon = (\beta/6)^{pt}$. Let $G'' = G' - V_0$, and let R be the Reduced Graph of G'' with parameters ε, k, d. It is easy to see that

$$\frac{e(R)}{k^2/2} \geq \frac{e(G'')}{n^2/2} > 1 - \frac{1}{p-1}.$$

Thus, by Turán's theorem, R contains a p-clique. Hence the Key Lemma guarantees that G_n contains $K_p(t,\ldots,t)$ provided n is large in terms of t. ∎

It is easy to see that we, in fact, proved the following stronger statement. (See also Frankl-Pach [63], and [116].)

Theorem 2.8 (Number of copies of H). *Let H be a graph with h vertices and chromatic number p. Let $\beta > 0$ be given and write $\varepsilon = (\beta/6)^h$. If n is large enough and a graph G_n has*

$$e(G_n) > \left(1 - \frac{1}{p-1} + \beta\right)\frac{n^2}{2}$$

then

$$\|H \to G_n\| > \left(\frac{\varepsilon n}{M(\varepsilon)}\right)^h.$$

It is interesting to contrast this with the following peculiar fact observed by Füredi. If a graph has few copies of a sample graph (e.g. few triangles), then they can all be covered by a few edges:

Theorem 2.9 (Covering copies of H). *For every $\beta > 0$ and sample graph H there is a $\gamma = \gamma(\beta, H) > 0$ such that if G_n is a graph with at most $\gamma n^{v(H)}$ copies of H, then by deleting at most βn^2 edges one can make G_n H-free.*

Proof. Write $h = v(H)$ and $\varepsilon = (\beta/3)^h$, and select $\gamma = (\varepsilon/M(\varepsilon))^h$. Assume (without loss of generality) that n is large enough, and apply the Degree Form of the Regularity Lemma with $d = \beta$ and, as before, let G'' be $G' - V_0$. We claim that **the graph G'' is H-free.** Let R be the Reduced Graph of G''. If G'' contained a copy of H, then R itself would have to contain either H or at least a graph H' such that $H \subset H'(h)$. But then, by the Key Lemma, we would have

$$\|H \to G_n\| > (\varepsilon m)^h > (\varepsilon n/k)^h \geq (\varepsilon n/M(\varepsilon))^h = \gamma n^h;$$

a contradiction. ∎

The above mentioned theorems can be proved directly without the Regularity Lemma, e.g. using sieve-type formulas, see [91, 92, 53, 21].

3 Early applications

Among the first graph theoretical applications, the Ramsey-Turán theorem for K_4 and the $(6,3)$-theorem of Ruzsa and Szemerédi were proved using the Old Szemerédi Lemma.

3.1 The $(6,3)$-problem

The $(6,3)$-problem is a special hypergraph extremal problem: Brown, Erdős and Sós asked for the determination of the maximum number of hyperedges an r-uniform hypergraph can have without containing ℓ hyperedges the union of which is at most k. One of the simplest cases they could not settle was this $(6,3)$-problem.

Theorem 3.1 (The $(6,3)$-theorem, Ruzsa-Szemerédi 1976 [108]). *If H_n is a 3-uniform hypergraph on n vertices not containing 6 points with 3 or more triangles, then $e(H_n) = o(n^2)$.*

It is easy to see that this theorem is equivalent to the following. (A matching M in G is induced if the only edges of G connecting vertices of M are those of M, i.e. "no cross edges".)

Theorem 3.2 (Induced matchings). *If G_n is the union of n induced matchings, then $e(G_n) = o(n^2)$.*

(The condition can be reformulated by saying that the edges of G_n are T-colored so that every path $P_4 \subset G_n$ is 4-colored. Determine the maximum of $e(G_n)$. Such problems were investigated among others by Burr, Erdős, Graham and Sós in [23], Burr, Erdős, Frankl, Graham and Sós in [22], and the analogous problems for C_5 were solved by Erdős and Simonovits in [51].)

It is interesting to note the following relation of the induced matching theorem (or the $(6,3)$-theorem) to $r_3(n)$ - the length of the longest sequence of integers up to n not containing a three-term arithmetic progression:

If $f(k,n)$ is the maximum number of edges an n-graph can contain if it is the union of k induced matchings, then $r_3(n) \le f(n,5n)/n$. Indeed, let $R = r_3(n)$, and let $a_1, \dots, a_R\,(\le n)$ be a maximum length sequence without a three-term arithmetic progression. Define the bipartite graph $G_{5n} = (A, B, E)$ as follows. $A = [2n]$, $B = [3n]$, and

$$E \subset A \times B, \quad E = \{(x + a_i, x + 2a_i) : x \in [n],\, i \in [R]\}.$$

Then G_{5n} has exactly Rn edges, and it is the union of the n matchings $M_x = \{(x+a_i, x+2a_i) : i \in [R]\}$. It remains to note the simple fact that the matchings M_x are induced in G_{5n}.

Thus, the estimate

$$f(k,n) < 2\varepsilon n^2 + k\varepsilon n \quad \text{for all large enough } n$$

proven below gives perhaps the simplest proof so far for Roth's theorem $r_3(n) = o(n)$. (Frankl and Rödl think that perhaps the general $r_k(n) = o(n)$ theorem also has a similar proof, where, for some extremal hypergraph problem EXT-PR(k) we have an upper bound $o(n^\ell)$ and a lower bound $cn^{\ell-1}r_k(n)$. They claim that this program works for proving $r_4(n) = o(n)$.)

Proof. We prove the following statement. Let $\varepsilon > 0$ be arbitrary and $n \ge 2M(\varepsilon)/\varepsilon^2$. If G_n is the union of k induced matchings, then $e(G_n) < 2\varepsilon n^2 + k\varepsilon n$.

Indeed, let us apply the Degree Form of the Regularity Lemma with parameter $d = 2\varepsilon$, and let $G'' = G' - V_0$. We claim that **any induced matching in G'' has at most εn edges.**

Let IM be an induced matching in G'', and write $U = V(IM)$ for the vertex set of IM, and $U_i = U \cap V_i$. Define $I = \{i : |U_i| > \varepsilon |V_i|\}$, and set $L = \cup_{i \in I} U_i$ and $S = U \setminus L$. Clearly $|S| \le \varepsilon n$. Hence, if we had $|U| > 2\varepsilon n$, then we would have $|L| > |U|/2$, and thus there would exist two vertices $u, v \in L$ adjacent in IM. Let $u \in V_i$ and $v \in V_j$. We would thus have an edge between V_i and V_j in the reduced graph R of G'', and hence a density more than 2ε between them. The sets U_i and U_j, being of size larger than εm each, would have a density more than ε between them. This means more than $\varepsilon |U_i||U_j| \ge \min\{|U_i|, |U_j|\}$ edges; a contradiction with IM being induced. ∎

(Since the function $M(\varepsilon)$ grows increadibly fast, this would only give an upper bound $r_3(n) = O(n/\log^* n)$, much weaker than Roth's $r_3(n) = O(n/\log\log n)$, let alone the often conjectured $r_3(n) = O(n/\log n)$. The best known upper bound is due to Heath-Brown [78] and to Szemerédi [122] improving Heath-Brown's result, according to which $r_3(n) \le O(n/\log^{1/4-\varepsilon} n)$.)

3.2 Applications in Ramsey-Turán theory

Theorem 3.3 (Ramsey-Turán for K_4, Szemerédi 1972 [119]). *If G_n contains no K_4 and only contains $o(n)$ independent vertices, then $e(G_n) < \frac{1}{8}n^2 + o(n^2)$.*

Proof. (Sketch) The proof is based on the following three simple statements (stated informally first) which we will not prove.

[1] If $\alpha(G) < t(G)$ then $K_3 \subset G$ (This is trivial.)
[2] If G_n is an ε-regular pair with $d(G_n) > 1/2$ and $\alpha = o(n)$, then $K_4 \subset G_n$.
[3] If G_n is an ε-regular triangle with $\alpha = o(n)$, then $K_4 \subset G_n$.

The precise forms of [2] and [3] are as follows. A, B, C denote disjoint sets.

[2] Let $G = (A \cup B, E)$, $|A| = |B| = m$, and assume that
(i) $G(A, B)$ is ε-regular with density at least $\beta + \varepsilon$, with some $\beta > 1/2$, $\varepsilon > 0$,
(ii) $\alpha(G) \le (2\beta - 1)m$.
Then $K_4 \subset G$.

[3] Let $G = (A \cup B \cup C, E)$, $|A| = |B| = |C| = m$, and assume that
(i) $G(A, B)$ is ε-regular with density at least $\beta + \varepsilon$, with some $\beta \ge \varepsilon > 0$, and the same holds for the pairs $(A, C), (B, C)$,
(ii) $\alpha(G) \le \beta^2 m$.
Then $K_4 \subset G$.

Now the proof of Theorem 3.3 goes as follows. Let

$$e(G_n) > (1/8 + 4\varepsilon)n^2, \; \alpha(G_n) \le \frac{\varepsilon^2}{M(\varepsilon)} n - 1, \quad \text{and} \quad n \ge M(\varepsilon)/\varepsilon.$$

Claim. $K_4 \subset G_n$.

Apply the Degree Form with parameters $d = 2\varepsilon$. Let $G'' = G' - V_0$ be the usual pure graph with Reduced Graph R. We have $e(G'') > (1/8 + \varepsilon)n^2$.

Also note that

$$\alpha(G_n) < \varepsilon^2 \left(\frac{n}{M(\varepsilon)} - 1 \right) \le \varepsilon^2 \left(\frac{n}{k} - 1 \right) < \varepsilon^2 m.$$

We will use the fact that we did not kill any edges in regular pairs of density greater than $\beta + \varepsilon = 2\varepsilon$ (we can't afford decreasing α even within these pairs!), and the edges inside clusters will be put back later on.

Case I. If more than $k^2/4$ edges in R are present, then, by Turán's theorem, R has a triangle. We can use [3] (with $\beta = \varepsilon$) to show that $K_4 \subset G''$.

Case II.

$$\sum_{1 \le i < j \le k} d(V_i, V_j) = e(G'')/m^2 \ge e(G'')k^2/n^2 > (1/8 + \varepsilon)k^2.$$

Hence, if at most $k^2/4$ of these densities are non-zero, then their average is greater than $d = 1/2 + 4\varepsilon$. Thus, at least one of them has a density greater than d. Let H be the graph consisting of this regular pair with the edges inside the two clusters put back. To show that $K_4 \subset G''$ we can apply [2] to H with $\beta = d - \varepsilon = 1/2 + 3\varepsilon$, since

$$\alpha(H) \le \alpha(G_n) < \varepsilon m < 6\varepsilon m = (2\beta - 1)m. \qquad \blacksquare$$

Remark. Most people believed that in Theorem 3.3 the upper bound $n^2/8$ can be improved to $o(n^2)$. To their surprise, in 1976 Bollobás and Erdős [15] came up with an ingenious geometric construction that showed that the constant 1/8 in the theorem is best possible. That is, they show the existence of a graph sequence (H_n) for which

$$K_4 \not\subset H_n, \quad \alpha(G_n) = o(n) \quad \text{and} \quad e(H_n) > \frac{n^2}{8} - o(n^2).$$

Remark. A typical feature of the application of the regularity lemma can be seen above, namely that we do not distinguish between $o(n)$ and $o(m)$, since the number k of clusters is bounded and $m \approx n/k$.

Remark. The problem of determining $\max e(G_n)$ under the condition

$$K_p \not\subset G_n \quad \text{and} \quad \alpha(G_n) = o(n)$$

is much easier for odd p than for even p. A theorem of Erdős and Sós describing the odd case was a starting point of the so-called **Theory of Ramsey-Turán problems**. The next important contribution was just the above-mentioned theorem of Szemerédi (and then the counterpart due to Bollobás and Erdős). Finally the paper of Erdős, Hajnal, Sós and Szemerédi [49] completely solved the problem for *all even* p by generalizing the above Szemerédi and Bollobás-Erdős theorems. It again used the Regularity Lemma. There are many related Ramsey-Turán theorems; we refer the reader to [47] and [48]. The very first Ramsey-Turán type problem can be found in the paper [117] of Vera Sós.

3.3 Other early applications

There are many applications where the Regularity Lemma is only as good as the old version. Perhaps one of the first applications where the new version was used is the paper of Bollobás, Erdős, Simonovits and Szemerédi [16], where various theorems on extremal graph problems with **large forbidden graphs** were discussed, and in two cases the new Regularity Lemma was used. One of these applications was an Erdős-Stone type application that we skip here. To state the other one we need a definition. Just as we defined the "blown-up" graph $H(t)$, we define $H(t_1, t_2, \dots, t_r)$ similarly by replacing the i-th vertex of H by t_i independent vertices, and replacing an edge (v_i, v_j) of H by the complete bipartite graph K_{t_i,t_j}.

Theorem 3.4. *Let t be an arbitrary natural number and c an arbitrary positive real number. Then there exists an n_0 such that if $n \geq n_0$, and G_n is a graph of order n then either G_n can be turned into a bipartite graph by deleting cn^2 edges or $C_{2k+1}(t) \subset G_n$ for some k satisfying $2k+1 < 1/c$.*

Remark. In the above theorem it is not too important if we use C_{2k+1} or $C_{2k+1}(t)$: most applications of Szemerédi Lemma are such that whenever we can ensure the occurence of a small subgraph L then we can also ensure the occurence of the blown-up graph $L(t)$. Similarly, in most applications of the Regularity Lemma it does not make any difference if we exclude $L \subset G_n$ or if we replace this condition by the weaker one that G_n contains at most $o(n^{v(L)})$ copies of L.

Remark. Theorem 3.4 has two proofs in [16]: one with and one without the Regularity Lemma. In many cases, the application of the Regularity Lemma makes things transparent but the same results can be achieved without it equally easily. One would like to know when one can replace the Regularity Lemma with "more elementary" tools and when the application of the Regularity Lemma is unavoidable. The basic experience is that when in the conjectured extremal graphs for a problem the densities in the Szemerédi partition are all near to 0 or 1, then the Szemerédi lemma can probably be eliminated. On the other hand, if these densities are strictly bounded away from 0 and 1 then the application of the Szemerédi lemma is typically unavoidable.

3.4 Generalized random graphs

We already mentioned that in a sense the Regularity Lemma says that all graphs can be approximated by generalized random graphs. The following observation was used in the paper of Simonovits and Sós [116] to characterize quasi-random graphs.

Theorem 3.5. *Let $\delta > 0$ be arbitrary, and let $V_0, V_1, \dots, V_k$ be a Szemerédi partition of an arbitrary graph G_n with $\varepsilon = \delta^2$ and each cluster size less than δn. Let Q_n be the random graph obtained by replacing the edges joining the classes V_i and V_j (for all $i \neq j$) by independently chosen random edges of probability $p_{i,j} := d(V_i, V_j)$, and let H be any graph with ℓ vertices. If $n \geq n_0$, then*

$$\|H \to Q_n\| - C_\ell \delta n^\ell \leq \|H \to G_n\| \leq \|H \to Q_n\| + C_\ell \delta n^\ell.$$

almost surely, where C_ℓ is a constant depending only on ℓ.

3.5 Building small induced subgraphs

While the reduced graph R of G certainly reflects many aspects of G, when discussing induced subgraphs the definition should be changed in a natural way. Given a partition $V_1, \dots, V_k$ of the vertex-set V of G and positive parameters ε, d, we define the induced reduced graph as the graph whose vertices are the clusters $V_1, \dots, V_k$ and V_i and V_j are adjacent if the pair (V_i, V_j) is ε-regular in G with density *between* d and $1 - d$. Then the following analogue of the Key Lemma (stated in a less quantitative manner) holds.

Theorem 3.6. *If the induced reduced graph of G contains an induced subgraph H, then so does G, provided that ε is small enough in terms of H. G even contains an induced copy of $H(r)$ provided that ε is small enough in terms of H and r.*

Below we will describe an application of the regularity lemma to ensure the existence of small induced subgraphs of a graph, not by assuming that the graph has many edges but by putting some condition on the graph which makes its structure randomlike, fuzzy.

Definition 3.7. *A graph $G = (V, E)$ has the property (γ, δ, σ) if for every subset $S \subset V$ with $|S| > \gamma|V|$ the induced graph $G(S)$ satisfies*

$$(\sigma - \delta)\binom{|S|}{2} \le e(G(S)) \le (\sigma + \delta)\binom{|S|}{2}.$$

Theorem 3.8 (Rödl 1986 [105]). *For every positive integer k and every $\sigma > 0$ and $\delta > 0$ such that $\delta < \sigma < 1 - \delta$ there exists a γ and a positive integer n_0 such that every graph G_n with $n \ge n_0$ vertices satisfying the property (γ, δ, σ) contains all graphs with k vertices as induced subgraphs.*

Rödl also points out that "this theorem yields an easy proof (see [94]) of the following generalization of a Ramsey theorem first proved in [31, 46] and [104]:

Theorem 3.9. *For every graph L there exists a graph H such that for any 2-coloring of the edges of H, H must contain an induced monochromatic L.*

The next theorem of Rödl answers a question of Erdős [11, 39].

Theorem 3.10. *For every positive integer k and positive σ and γ there exists a $\delta > 0$ and a positive integer n_0 such that every graph G_n with at least n_0 vertices having property (γ, δ, σ) contains all graphs with k vertices as induced subgraphs.*

(Erdős asked if the above theorem holds for $\frac{1}{2}, \delta, \frac{1}{2}$ and K_k.)

The reader later may notice the analogy and the connection between this theorem and some results of Chung, Graham and Wilson on quasi-random graphs (see Section 10).

3.6 Diameter-critical graphs

Consider all graphs G_n of diameter 2. The minimum number of edges in such graphs is attained by the star $K(1, n-1)$. There are many results on graphs of diameter 2. An interesting subclass is the class of 2-diameter-critical graphs. These are minimal graphs of diameter 2: deleting any edge we get a graph of diameter > 2. C_5 is one of the simplest 2-diameter-critical graphs. If H is a 2-diameter-critical graph, then $H(a_1, \dots, a_k)$ is also 2-diameter-critical. So $T_{n,2}$, and more generally of $K(a, b)$, are 2-diameter-critical. Independently, Murty and Simon (see in [24]) formulated the following conjecture:

Conjecture 3.11. *If G_n is a minimal graph of diameter 2, then $e(G) \le \lfloor n^2/4 \rfloor$. Equality holds if and only if G_n is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

Füredi used the Regularity Lemma to prove this.

Theorem 3.12 (Füredi 1992 [66]). *Conjecture 3.11 is true for $n \ge n_0$.*

Here is an interesting point: Füredi did not need the whole strength of the Regularity Lemma, only a consequence of it, the $(6, 3)$-theorem.

4 Building large subgraphs

Most of the proofs presented here and in the subsequent sections will be simplified by the application of the Key Lemma. While this is something of an *anachronism* the original proof ideas are not lost, they are just basically summarized in the Key Lemma.

4.1 Packing with small graphs

Theorem 4.1 (Alon-Yuster 1992 [6]). *For any $\alpha > 0$ and H there is an n_0 such that*

$$n \geq n_0, \quad \delta(G_n) > \left(1 - \frac{1}{\chi(H)} + \alpha\right) n$$

imply that there are $(1-\alpha)n/|V(H)|$ vertex-disjoint copies of H in G_n.

In other words, G_n can be almost completely covered by copies of H.

Proof. Let $r = \chi(H)$, and apply the Degree Form with $d = \alpha/2$ and a very small ε to get the usual pure graph G'' with reduced graph R. Then apply the Hajnal and Szemerédi theorem (Theorem 2.7) for the graph R. Thus R is covered by "ε-regular r-cliques".

Let $h = v(H)$ and notice that $K_r(h)$ contains the union of r vertex-disjoint copies of H. (That is, we could assume that H has the same number of vertices in each of the r color classes.) The Key Lemma (and the remark after that) implies that an ε-regular r-clique with density greater than d on each edge can be covered almost perfectly by vertex-disjoint copies of $K_r(h)$ (and hence those of H), since the union of vertex-disjoint copies of $K_r(h)$ has bounded degree. ∎

Recently, Alon and Yuster [7] improved on their own result by showing that the tiling of G_n with copies of H is perfect (provided, of course, that $v(H)$ divides n). For their beautiful conjecture (that even $\alpha = 0$ works) see Section 6.

4.2 Large subgraphs with bounded degrees

The following theorem is implicit in Chvátal-Rödl-Szemerédi-Trotter 1983 [27] (according to Alon, Duke, Leffman, Rödl and Yuster [3]).

Theorem 4.2. *For any $\Delta, \beta > 0$ there is a $c > 0$ such that if $e(G_n) > \beta n^2$, then G_n contains as subgraphs all* **bipartite** *graphs H with $|V(H)| \leq cn$ and $\Delta(H) \leq \Delta$.*

Proof. It is enough to pick one single ε-regular pair (with a sufficiently small ε) from a regular partition of the host graph G_n, and then apply the Key Lemma. ∎

The next theorem is central in Ramsey theory. It says that the Ramsey number of a bounded degree graph is linear in the order of the graph.

Theorem 4.3 (Chvátal-Rödl-Szemerédi-Trotter 1983 [27]). *For any $\Delta > 0$ there is a $c > 0$ such that if G is any n-graph, and H is any graph with $|V(H)| \le cn$ and $\Delta(H) \le \Delta$, then either $H \subset G$ or $H \subset \overline{G}$.*

Proof. Let $r = \chi(H)$, and let us start again with a regular partition of G_n (with a small ε). Throw away all edges in non-regular pairs and form the Reduced Graph R of the leftover. Color an edge of R BLUE if the density in G_n between the corresponding clusters is at least $1/2$, otherwise color it RED. The application of the following trivial observation will lead to either a BLUE r-clique or to a RED r-clique in R.

Fact. *For every r there is an $\varepsilon > 0$ and an n_0 such that if we two-color the edges of a graph with $n \ge n_0$ vertices and at least $(1-\varepsilon)\binom{n}{2}$ edges, then it contains a monochromatic r-clique.*

An application of the Key Lemma completes the proof of Theorem 4.3. ∎

5 Embedding trees

So far all embedding questions we discussed dealt with embedding bounded degree graphs H into dense graphs G_n. General Ramsey theory tells us that this cannot be relaxed substantially without putting strong restrictions on the structure of the graph H. (Even for bipartite H, the largest complete bipartite graph $K_{\ell\ell}$ that a dense graph G_n can be expected to have is for $\ell = O(\log n)$.) A frequently used structural restriction on H is that it is a tree (or a forest). Under this strong restriction even very large graphs H can be embedded into dense graphs G_n.

The two extremal cases are when H is a large star, and when H is a long path. Both cases are precisely and easily handled by classical extremal graph theory (Turán theory or Ramsey theory). The use of the Regularity Lemma makes it possible, in a sense, to reduce the case of general trees H to these two special cases by splitting the tree into "long" and "wide" pieces. After an application of the Regularity Lemma one applies, as always, a classical graph theorem, which in most cases is the König-Hall matching theorem, or the more sophisticated Tutte's theorem (more precisely, the Gallai-Edmonds decomposition).

5.1 The Erdős-Sós conjecture for trees

Conjecture 5.1 (Erdős-Sós 1963 [54]). *Every graph on n vertices and more than $(k-1)n/2$ edges contains, as subgraphs, all trees with k edges.*

In other words, if the number of edges in a graph G forces the existence of a k-star, then it also guarantees the existence of any other subtree with k edges. The theorem is known for k-paths (Erdős-Gallai 1959 [44]).

Remark. The assertion is trivial if we put up with loosing a factor of 2: If G has average degree at least $2k > 0$, then it has a subgraph G' with $\delta(G') > k$, but then the greedy algorithm guarantees that G' contains all k-trees.

Here we formulate the following result of Ajtai, Komlòs and Szemerédi.

Theorem 5.2 (Erdős-Sós conjecture - approximate form 1991 [2]). *For every $\varepsilon > 0$ there is a threshold k_0 such that the following statement holds for all $k \geq k_0$: Every graph with average degree more than $(1+\varepsilon)k$ contains, as subgraphs, all trees with k edges.*

It is important to note that the authors' 1991 manuscript contains only the "dense case", that is, when $n \leq Ck$. The "sparse case" needs a modified form of the Regularity Lemma that is not as compact and as generally applicable as the original Regularity Lemma.

5.2 The Loebl conjecture

In their paper about graph discrepancies P. Erdős, Z. Füredi, M. Loebl and V. Sós [43] reduced some questions to the following conjecture of Loebl:

Conjecture 5.3 (Martin Loebl). *If G is a graph on n vertices, and at least $n/2$ vertices have degrees at least $n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges.*

In fact, the following approximation result proved by Ajtai, Komlós and Szemerédi [1] was enough for [43].

Theorem 5.4 (Loebl conjecture - approximate form, [1]). *For every $\varepsilon > 0$ there is a threshold n_0 such that for all $n \geq n_0$, if G_n has at least $(1+\varepsilon)n/2$ vertices of degre at least $(1+\varepsilon)n/2$, then G_n contains, as subgraphs, all trees with at most $n/2$ edges.*

Note that Conjecture 5.3 has a strong similarity to the celebrated Erdős-Sós conjecture, though it is probably much easier. The main tool used for the proof of this approximate form was again the Regularity Lemma. J. Komlós and V. Sós generalized Loebl's conjecture for trees of any size. It says that any graph G contains all trees whose number of edges do not exceed the medium degree of G.

Conjecture 5.5. *If G is a graph on n vertices, and at least $n/2$ vertices have degrees greater than or equal to k, then G contains, as subgraphs, all trees with k edges.*

In other words, the condition in the Erdős-Sós conjecture that the *average* degree be greater than $k-1$, would be replaced here with a similar condition on the *median* degree.

The following example shows that the conjecture - if true - is close to best possible.

Let $n = k+1$ and partition the vertex-set into parts V_1, V_2, where $|V_1| = \lfloor \frac{k-1}{2} \rfloor$. Make all edges within V_1 and also between V_1 and V_2. While all vertices in V_1 have degree k, the graph does not contain a path of length k. Use disjoint copies of this graph to get

$$\liminf_{n\to\infty} \frac{1}{n} f_k(n) \geq \lfloor \frac{k-1}{2} \rfloor \,/\, (k+1),$$

where $f_k(n)$ is the maximum number of vertices of degree k or more an n-graph can have without having *all* trees of size k as subgraphs. This general conjecture is probably not much easier than the Erdős-Sós conjecture.

Komlós and Szemerédi announced that they can prove an approximate version of Conjecture 5.5. It needs an auxiliary lemma they already developed for attacking the Erdős-Sós conjecture, and again a sparse form of the Regularity Lemma.

5.3 A Bollobás conjecture on spanning trees

Definition 5.6. *Given a set of graphs $G_1, G_2, \ldots, G_\ell$, we say that $G_1, G_2, \ldots, G_\ell$ can be* **packed** *into G if we can find embeddings φ_i of G_i into G such that the edge-sets $\varphi_i(E(G_i))$ are pairwise disjoint. If $G = K_n$, the complete graph on n vertices, then we simply say that there is a packing of $G_1, G_2, \ldots, G_\ell$.*

The notion of packing plays an important role in the investigation of computational complexity of graph properties among other things. Thus it is not surprising that in recent research literature there is considerable interest in packing-type results and problems (see e.g. [11, 14, 73, 109]).

Bollobás [11] conjectured that trees of bounded degrees can be embedded into graphs of degree roughly $n/2$. It was recently proved by J. Komlós, G. N. Sárközy and E. Szemerédi.

Theorem 5.7 (Komlós-Sárközy-Szemerédi 1993 [84]). *For every $\varepsilon > 0$ and Δ there is a threshold n_0 such that the following statement holds for all $n \geq n_0$: If T is a tree of order n and maximum degree Δ, and G_n has minimum degree at least $(1+\varepsilon)n/2$, then T is a subgraph of G_n.*

The theorem is actually true even for trees of maximum degree $cn/\log n$ with a small enough $c > 0$, and this is sharp. We remark that Sárközy gave an NC^4 algorithm that actually exhibits such a tree-embedding. His algorithm also finds Hamiltonian cycles in so-called δ-Pósa graphs: n-graphs in which the degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ satisfies the following Pósa type condition:

$$d_k \geq \min\left\{k + \delta n, \frac{n}{2}\right\} \quad \text{for} \quad 1 \leq k \leq n.$$

Note that the critical point in the Bollobás conjecture is that the tree is a spanning tree. For somewhat smaller trees everything is much simpler. The following is an easy exercise in the use of the Regularity Lemma: For each $\varepsilon > 0$ there is an $\alpha > 0$ and a threshold n_0 such that if $\delta(G_n) \geq n/2$, where $n \geq n_0(\varepsilon)$, then G_n contains as subgraphs all trees of order at most $(1-\varepsilon)n$ with maximum degree at most αn.

The Bollobás conjecture was proved using the Regularity Lemma and the following interesting lemma about disjoint connections.

Lemma 5.8 (Komlós-Sárközy-Szemerédi 1993 [84]). *Let G contain $4n$ vertices: $V = V_1 + V_2 + V_3 + V_4$ with each $|V_i| = n$. Assume that, for $i = 1, 2, 3$, the edges between V_i and V_{i+1} form an ε-regular pair of density at least $d > 0$. If $\varepsilon \le \varepsilon_0(d)$ and $n \ge n_0(d)$, and if φ is any bijection from V_1 to V_4, then there is a collection of n pairwise vertex-disjoint paths of order 4 connecting v with $\varphi(v)$ for all $v \in V_1$.*

This last lemma combined with the Key Lemma were the germs of the general tool formulated below as Blow-up Lemma.

6 Bounded degree spanning subgraphs

This is probably the most interesting class of embedding problems. Here the proofs (when they exist) are too complicated to quote here, but they follow a general pattern. When embedding H to G (they have the same order now!), we first prepare H by chopping it into (a constant number of) small pieces, then prepare the host graph G by finding a regular partition of G, throw away the usual atypical edges, and define the reduced graph R. Then typically we apply to R the matching theorem (for bipartite H) or the Hajnal-Szemerédi theorem (for r-partite H). At this point, we make an assignment between the small pieces of H and the "regular r-cliques" of the partitioned R. There are two completely different problems left. Make the connections between the r-cliques, and embed a piece of H into an r-clique. The first one is sometimes easy, sometimes very hard, but there is no general recipe to apply here. The second part, however, can typically be handled by referring to the so-called Blow-up Lemma – a new general purpose embedding tool discussed below.

6.1 The Bollobás-Eldridge conjecture

The next conjecture is perhaps the most beautiful one in the field of packing.

Conjecture 6.1 (Bollobás-Eldridge 1978 [13, 14]). *If $(\Delta(G_n)+1)(\Delta(G'_n)+1) \le n+1$ then G_n and G'_n can be packed.*

Note that the celebrated Hajnal-Szemerédi theorem is a special instance of this conjecture, namely when G'_n is a union of cliques (it was stated in an earlier section in a complementary form).

The particular case when G'_n has maximum degree 2 was separately conjectured by Sauer and Spencer in 1978 [109] (disjoint union of cycles) and was recently solved for large n by Noga Alon and Eldar Fischer [4]. (Note that the hardship again comes from the fact that the graph H to be embedded is spanning. Embedding into a G_n with $\delta(G_n) \ge (2/3)n$ unions of cycles with total order $(1-\varepsilon)n$ is a routine exercise in the use of the Regularity Lemma.)

The Erdős-Stone theorem, as well as the Alon-Yuster theorem (see below) suggest that the critical parameter should be $\chi(G'_n)$ rather than $\Delta(G'_n)$. Thus the following conjecture would be natural. Let $\varepsilon > 0$ be given, let G_1 and G_2 be two n-graphs, and let G_1 have bounded degrees. If $\chi(G_1)\Delta(G_2) \leq (1-\varepsilon)n$, then G_1 and G_2 can be packed. Or using the complementary form (embedding rather than packing) with $H = G_1$ and $G = \overline{G_2}$: If $\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \varepsilon\right) n$, then $H \subset G$. Unfortunately, this is false even for $\chi(H) = 2$ as the following example shows: Let H be a random bipartite graph and G be the union of two equal size cliques sharing only εn vertices. Since H is an expander but G is not, we cannot have $H \subset G$. The narrow communication bottleneck between the two cliques of G suggests a technical condition that may help: small bandwidth $w(H)$.

Conjecture 6.2 (Bollobás-Komlós 1994). *For each $\varepsilon, \Delta > 0$ and r there is an $\alpha > 0$ and an n_0 such that if $v(H) = v(G) = n \geq n_0$, $\chi(H) \leq r$, $\Delta(H) \leq \Delta$, $w(H) \leq \alpha n$, and*

$$\delta(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right) n,$$

then H is a subgraph of G.

6.2 The Pósa-Seymour conjecture

Paul Seymour conjectured in 1973 that any graph G of order n and minimum degree at least $\frac{k}{k+1}n$ contains the k-th power of a Hamiltonian cycle. For $k = 1$, this is just Dirac's theorem. For $k = 2$, the conjecture was made by Pósa in 1962. Note that the validity of the general conjecture would imply the notoriously hard Hajnal-Szemerédi theorem.

The following approximate version was recently proved.

Theorem 6.3 (Pósa-Seymour conjecture - approximate form, 1994 [85]). *For any $\varepsilon > 0$ and positive integer k there is an n_0 such that if G has order $n \geq n_0$ and minimum degree at least $\left(1 - \frac{1}{k+1} + \varepsilon\right) n$, then G contains the k-th power of a Hamilton cycle.*

The authors of the last theorem announced that they can also prove the precise Pósa conjecture. For partial results, see the papers of Jacobson (unpublished), Faudree, Gould, Jacobson and Schelp [61], Häggkvist (unpublished), Fan and Häggkvist [57], Fan and Kierstead [58], Faudree, Gould and Jacobson [60], and Fan and Kierstead [59]. Fan and Kierstead also announced a proof of the Pósa conjecture if the Hamilton cycle is replaced by Hamilton path. (Noga Alon observed that this already implies the Alon-Fischer result mentioned in the previous subsection, for the square of a Hamilton path contains all unions of cycles.) We do not detail the exact statements in these papers, since none of the papers employ the Regularity Lemma.

6.3 The Alon-Yuster conjecture

The beautiful conjecture of Alon and Yuster 1992 [6] generalizes the celebrated Hajnal-Szemerédi theorem from covering with cliques to covering with copies of an arbitrary (fixed) graph H. A solution has been announced recently:

Theorem 6.4 (Komlós-Sárközy-Szemerédi 1995 [87]). *For every graph H there is a constant K such that*

$$\delta(G_n) \geq \left(1 - \frac{1}{\chi(H)}\right) n$$

implies that there is a union of vertex-disjoint copies of H covering all but at most K vertices of G_n.

A simple example in [6] shows that $K = 0$ cannot always be achieved even when $v(H)$ divides $v(G)$. Erdős and Faudree conjectured that for $H = K_{2,2}$ a perfect covering is possible, that is, if n is divisible by 4, and G_n has minimum degree $n/2$, then G_n can be perfectly tiled by 4-cycles. [87] also contains a proof of that for large enough n.

The proof goes along the following lines. Let $r = \chi(H)$ and apply first the Regularity Lemma and then the Hajnal-Szemerédi theorem for the reduced graph R to obtain a covering of most of the vertices with super-regular r-cliques with equal color classes. Then the leftover εn vertices are distributed among these regular r-cliques as evenly as possible. This may not be possible completely evenly; the color classes may differ with $\varepsilon' n$. At this point, the Alon-Yuster conjecture easily follows from the Blow-up Lemma in the (moderately interesting) case when the color classes of H are not all equal. For H with equal color classes, the somewhat uneven distribution of the vertices among the r clusters in a regular r-clique may be a problem. But one can show that either there is a partition of G_n into regular r-cliques with perfectly equal clusters and a constant number of leftover vertices (and thus the Blow-up Lemma implies the Alon-Yuster conjecture), or G_n has a very special structure in that it contains an r-partite subgraph G' with equal color classes and at least $(1-\varepsilon)n$ vertices and with minimum degree $\delta(G') > (1 - 1/r - \varepsilon)n$. It is not hard to see that such special graphs also satisfy the Alon-Yuster conjecture (even with $K = 0$).

6.4 The Blow-up Lemma

Several recent results exist about embedding spanning graphs into dense graphs. Some of the proofs use the following new powerful tool. It basically says that regular pairs behave as complete bipartite graphs from the point of view of embedding bounded degree subgraphs. Note that for embedding spanning subgraphs, one needs all degrees of the host graph large. That's why using regular pairs is not sufficient any more, we need super-regular pairs. The Blow-up Lemma plays the same role in embedding spanning graphs H into G as the Key Lemma played in embedding smaller graphs H (up to $v(H) < (1-\varepsilon)v(G)$).

The proof of the Blow-up Lemma starts with a probabilistic greedy algorithm, and then uses a König-Hall argument to finish the embedding. The proof of correctness is quite involved, and we will not present it here.

Theorem 6.5 (Blow-up Lemma – Komlós-Sárközy-Szemerédi 1994 [86]). *Given a graph R of order r and positive parameters δ, Δ, there exists an $\varepsilon > 0$ such that the following holds. Let $n_1, n_2, \ldots, n_r$ be arbitrary positive integers and let us replace the vertices of R*

with pairwise disjoint sets $V_1, V_2, \ldots, V_r$ *of sizes* $n_1, n_2, \ldots, n_r$ *(blowing up). We construct two graphs on the same vertex-set* $V = \cup V_i$*. The first graph* $\mathbf{R}$ *is obtained by replacing each edge* $\{v_i, v_j\}$ *of* R *with the complete bipartite graph between the corresponding vertex-sets* V_i *and* V_j*. A sparser graph* G *is constructed by replacing each edge* $\{v_i, v_j\}$ *with an* (ε, δ)*-super-regular pair between* V_i *and* V_j*. If a graph* H *with* $\Delta(H) \le \Delta$ *is embeddable into* $\mathbf{R}$ *then it is already embeddable into* G*.*

7 Weakening the Regularity Lemma

In a number of applications of the Regularity Lemma (especially the ones about bipartite graphs like Theorem 4.2) only one regular pair is used. Since the Regularity Lemma only guarantees that every n-graph G_n with cn^2 edges contains a regular pair of order at least $c'n$ where $c' = 1/tower(1/c)$ (the tower function of $1/c$) (and about the same density as that of G_n), in these situations a direct method is preferable that may not provide a full regular partition, but proves the existence of a regular pair with a much larger order. Such a method is the graph-functional method of Komlós 1991. In the next few subsections, we describe some applications of the functional method (a kind of moment method), and we will mention other variants of the regularity lemma later. While all these methods are weaker than the original Regularity Lemma of Szemerédi, they have the advantage of more manageable constants, and thus, more importantly, they can be applied to sparser graphs (e.g. n-graphs with $n^{1.9}$ edges).

7.1 The method of graph-functionals

Moment methods are standard tools in graph theory. The following special moments, called graph-functionals, were introduced by Komlós in 1991. They have the form

$$\psi(G) = \psi_1(d(G))\psi_2(v(G)),$$

where $d(G) = e(G)/\binom{v(G)}{2}$ is the density of G, and ψ_1 and ψ_2 are monotone increasing positive functions. For technical reasons, we also assume that $\psi_1(x)/x$ is monotone increasing and $\psi_2(x)/x$ is monotone decreasing. We often normalize these forms into one of the *standard forms*: $\psi(G) = \varphi(d(G))v(G)$ with an increasing φ, or $\psi(G) = t(G)\varphi(v(G))$ with a decreasing φ, where $t(G)$ is the average degree of G. The idea is that if the edge distribution in G is not uniform, then we may wish to replace G by a denser, but not much smaller subgraph H. The factor $\psi_2(v(G))$ guarantees that we do not choose too small H (e.g. an edge); it is a delicate balance that has to be set separately for every problem. The use of a graph functional is as follows:

- Given a graph G, select a subgraph H with maximal ψ: $\psi(H) = \max_{G' \subset G} \psi(G')$.
- Note that the order of H cannot be too small: $\psi_1(d(G))\psi_2(v(G)) \le \psi_1(d(H))\psi_2(v(H)) \le \psi_1(1)\psi_2(v(H))$ gives a lower bound on $v(H)$.

- Prove that every ψ-maximal graph has certain desirable properties.

The desirable properties will be similar to expanding properties, and are natural relaxations of regularity.

Given a class $\mathcal{C}$ of graphs, we say that $H \in \mathcal{C}$ is ψ-maximal within the class $\mathcal{C}$ if

$$\psi(H) = \max_{S \subset H, S \in \mathcal{C}} \psi(S).$$

When the class $\mathcal{C}$ is understood we often omit any reference to it and simply say H is ψ-maximal. It is worth noting that under the above conditions on the functions ψ_1, ψ_2, a ψ-maximal graph H is automatically t-maximal, too:

$$t(H) = \max_{S \subset H, S \in \mathcal{C}} t(S).$$

We start with two trivial lemmas.

Lemma 7.1. *Let $\psi(G) = v(G)\varphi(d(G))$, where $\varphi(x)$ is monotone increasing on $x \in [0,1]$. Then every graph $G \in \mathcal{C}$ contains a ψ-maximal subgraph $H \in \mathcal{C}$ satisfying*

$$\psi(H) \geq \psi(G), \quad d(H) \geq d(G), \quad v(H) \geq \psi(G)/\varphi(1).$$

Indeed, let

$$\psi(H) = \max_{G' \subset G, G' \in \mathcal{C}} \psi(G').$$

The lemma follows from the inequalities

$$\psi(G) = v(G)\varphi(d(G)) \leq v(H)\varphi(d(H)) \leq v(H)\varphi(1).$$

∎

Lemma 7.2. *Let $\psi(G) = t(G)\varphi(v(G))$, where $\varphi(x)$ is monotone decreasing and $\varphi(x) \geq 1$ for $x \geq 1$. Then every graph $G \in \mathcal{C}$ contains a ψ-maximal (as well as t-maximal) subgraph $H \in \mathcal{C}$ satisfying*

$$\psi(H) \geq \psi(G), \quad t(H) \geq t(G)/\varphi(1), \quad v(H) > \psi(G)/\varphi(1).$$

Indeed, let

$$\psi(H) = \max_{G' \subset G, G' \in \mathcal{C}} \psi(G').$$

The lemma follows from the inequalities

$$t(G) \leq t(G)\varphi(v(G)) \leq t(H)\varphi(v(H)) \leq t(H)\varphi(1) < v(H)\varphi(1).$$

The t-maximality is trivial. ∎

In the following few subsections we describe a couple of applications of this method.

7.2 Regular subgraphs

In 1992, at a workshop in Bielefeld, Erdős, Loebl and Sós started investigating a new area of Combinatorial Discrepancy Theory called *Graph Discrepancies.* Later Füredi and Ruzsa also joined these investigations (see [43]). The following related problem was asked by Vera Sós. Given a graph G_n with $e(G_n) > cn^2$, how large a regular subgraph can be found in G_n? The problem itself has many variants and a long history. Mostly people wanted to know how many edges are required to ensure a 3-regular, or more generally, a k-regular subgraph in a G_n. Denote the family of k-regular graphs by $\mathcal{L}_{k-reg}$. Erdős and Sauer [40] conjectured that

$$\mathrm{ex}(n, \mathcal{L}_{k-reg}) = O(n).$$

Alon, Friedland and Kalai [5] proved that every graph with maximum degree 5 and average degree bigger than 4 contains a 3-regular subgraph (and established some similar results for other degrees of regularity). Using this result Pyber proved [99] that

$$\mathrm{ex}(n, \mathcal{L}_{k-reg}) \le 32k^2 n \log n.$$

Then Pyber, Rödl and Szemerédi proved [100]

$$\mathrm{ex}(n, \mathcal{L}_{k-reg}) \ge cn \log\log n \quad \text{for some} \quad c > 0.$$

About dense graphs they note that an application of the Regularity Lemma and the matching theorem trivially show that for every $c > 0$ there is an $f(c) > 0$ such that $e(G_n) > cn^2$ implies that G_n contains a regular subgraph with $f(c)n^2$ edges. The example of the complete bipartite graph $K_{cn,(1-c)n}$ shows that $f(c) > c^2$ cannot be hoped for. The Regularity Lemma argument only gives something like $f(c) > 1/tower(1/c)$. Erdős asked if a polynomial lower bound for $f(c)$ can be found. In [88], Komlós and Sós provide such a bound by a simple application of the functional method (with $\psi(G) = v(G)d^r(G)$). In fact, they show the almost optimal $f(c) < c^2/\log^\rho(2/c)$. To get the first polynomial bound $f(c) < c^{4.82}$, let $r_0 = \log 2/(2\log 2 - \log 3) = 2.41$, and for $r > r_0$ let us write

$$c_r = 2 - \frac{3}{2}2^{1/r} > 0.$$

Then the following simple procedure gives the bound $f(d) \ge c_r d^{2r}$ for any $r > r_0$: Let $\mathcal{B}_n$ be be the class of all bipartite graphs with n vertices in each color classes, and let $\mathcal{B} = \cup_n \mathcal{B}_n$. For $G \in \mathcal{B}_m$ let us write $n(G) = m$. Fix an $r > r_0$, and for a graph $B \in \mathcal{B}$ define $\psi(B) = n(B)d^r(B)$, where $d(B) = e(B)/n^2(B)$ (bipartite density). We say that $H \in \mathcal{B}$ is ψ-maximal if

$$\psi(H) = \max_{\substack{H' \subset H \\ H' \in \mathcal{B}}} \psi(H').$$

Now to find a large regular subgraph from a dense graph G_n:

1. Select a bipartite subgraph $B \in \mathcal{B}$ of G_n that contains at least half the edges of G_n.

2. Select a subgraph $H \in \mathcal{B}$ of B that is ψ-maximal among all subgraphs of B within the class $\mathcal{B}$.

3. Apply the following two fairly simple lemmas (the first one is just Lemma 7.1).

Lemma 7.3. *Let $r > r_0$. If $B \in \mathcal{B}$ then G has a ψ-maximal subgraph (within the class $\mathcal{B}$) with $d(H) \geq d(B)$ and $n(H) \geq d^r(B)n(B)$.*

Lemma 7.4. *If $H \in \mathcal{B}$ is ψ-maximal (within $\mathcal{B}$) then H contains $c_r d(H)n(H)$ pairwise edge-disjoint complete one-factors.*

To get the tighter bound mentioned above, one needs some randomized versions of the method, and select a large matching of regular pairs. Note that this method provides fairly large regular subgraphs even for sparser n-graphs, e.g. with n-graphs with only $n^{1.9}$ edges.

7.3 Finding a larger regular pair

As mentioned above, often one single regular pair (or a large matching of regular pairs) is enough to select. The following theorem provides a regular pair having an order polynomial in the density and exponential in $1/\varepsilon$. (A repeated application gives a larger matching of such pairs. The decomposition described is Subsection 7.7 can also be used to provide a decent size regular pair, but not as large as the one guaranteed by the next theorem.)

Theorem 7.5 (Komlós 1991 [83]). *Let $\varepsilon \leq \varepsilon_0$ and $r = (3/\varepsilon)\log(1/\varepsilon)$. Then every $G \in \mathcal{B}$ contains an (ε, δ)-super-regular subgraph $H \in \mathcal{B}$ with*

$$\delta \geq d(G)/2 \quad \text{and} \quad n(H) \geq d^r(G)\, n(G).$$

Corollary 7.6. *Let $\varepsilon \leq \varepsilon_0$ and $r = (3/\varepsilon)\log(1/\varepsilon)$. If G_n is any n-graph with cn^2 edges, then G_n contains an (ε, δ)-super-regular subgraph $H \in \mathcal{B}$ with*

$$\delta \geq c \quad \text{and} \quad n(H) \geq (2c)^r \lfloor n/2 \rfloor.$$

The proof is again using the simple graph-functional $\psi(G) = v(G)d^r(G)$ but now with the large r defined in the theorem. We simply choose a ψ-maximal subgraph first, and then apply the following fairly easy lemma.

Lemma 7.7. *Let $H \in B$ be ψ_r-maximal with r defined above. Then H is (ε, δ)-super-regular with $\delta \geq d(H)/2$.*

7.4 Topological cliques in dense graphs

Here is another application where choosing one regular pair is sufficient, although that does not give the best result.

The **topological clique number** $tcl(G)$ of a graph G is the largest integer r such that G has a subgraph isomorphic to a subdivision of K_r, the complete graph on r vertices. A standard exercise in graph theory courses is the following simple theorem.

Theorem 7.8 (Erdős-Fajtlowicz [41]). *For most n-graphs G_n,*

$$c_1\sqrt{n} < tcl(G_n) < c_2\sqrt{n}.$$

(Bollobás and Catlin [12] improved this to $tcl(G_n) \sim \sqrt{2n}$ for most G_n.) The proof of the lower bound consists of picking k vertices $v_1, \ldots, v_k$ arbitrarily, and connecting them with (disjoint) paths of length at most 2 as follows. For each non-adjacent pair v_i, v_j, select a vertex from among their common neighbours, this way connecting them with a path of length 2. Since in most n-graphs these common neighbourhoods are of size about $n/4$, we can select an unused vertex every time, provided $\binom{k}{2} + k$ is less than about $n/4$ (or at least $k^2/4 < n/4$). The upper bound is equally simple.

The truth is, however, that the existence of a topological clique of size $c\sqrt{n}$ in a random n-graph is simply due to the fact that most graphs are dense, and **all dense n-graphs G_n have $tcl(G_n) > c\sqrt{n}$.**

Theorem 7.9. (*Komlós-Szemerédi 1994* [89, 90]) *For each $c > 0$ there is a $c' > 0$ such that $tcl(G_n) > c'\sqrt{n}$ for all graphs G_n with $e(G_n) > cn^2$.*

The proof of the existence of such a c' is fairly simple: We define some kind of expanders with the property that any two disjoint vertex sets of size cn are connected and all degrees are large. Then we show that these expanders have large topological subgraphs by using a greedy algorithm, and the simple fact that these expanders have a diameter at most 4. It remains to show that dense graphs have large expanders. This can be done naturally by using the Regularity Lemma to select a regular pair (and throw away a few vertices to get all degrees large). This was the way the following more general result was proved.

Theorem 7.10. (*Alon-Duke-Leffman-Rödl-Yuster 1993* [3]) *For each $c > 0$ there is a $c' > 0$ such that $e(G_n) > cn^2$ and $e(H) < c'n$ imply that H is a topological subgraph of G_n.*

The use of the Regularity Lemma is not really necessary here, since only the *existence of one single regular pair* in any dense graph is used. This existence can be shown without referring to the Regularity Lemma just by using direct computation, and this way one may get more reasonable constants (see Theorem 7.5). Hence the dependence of c' on c will not be something useless like $c' \approx 1/tower(1/c)$ (the tower function grows *real fast*), but "only" $c' \approx e^{-1/c}$. This seems to be quite an improvement. However, the true c' is proportional to $\sqrt{c}$, and thus another approach seems to be necessary. Ironically, returning to the Regularity Lemma is the solution. The proof of Theorem 7.9 uses the Regularity Lemma and gives a constant about $2\sqrt{c}$, which is within a factor 2 of the truth. Also, that proof implies that in the Alon-Duke-Leffman-Rödl-Yuster theorem the condition $e(H) < c'n$ can be relaxed to $v(H) < c'n$ and $e(H) < (2-\varepsilon)cn$ (here c' is, however, the ridiculously small c' obtained from the Regularity Lemma.)

7.5 Topological cliques in sparse graphs

The following question has obvious implications for simulating large complete network connections in sparse networks preferably using short paths. Let $f(t)$ be the largest integer such that every graph with average degree at least t has a topological clique with $f(t)$ vertices:

$$f(t) = \min\{tcl(G) : t(G) \geq t\},$$

where $t(G)$ is the average degree of the graph G. Determine, or estimate, the function $f(t)$. Mader [93], and independently Erdős and Hajnal [45], conjectured that

$$c_1\sqrt{t} < f(t) < c_2\sqrt{t},$$

that is, random dense graphs are the worst case. Mader's conjecture was first fully proved by Bollobás and Thomason [17] in 1994, followed in a few months by another proof of Komlós and Szemerédi [90]. The Bollobás-Thomason proof is direct, using sophisticated connectivity theory along the lines of some very recent results of Robertson and Seymour. They get the constant $c_1 = 1/\sqrt{512}$ (recently improved by them to $c_1 = 1/\sqrt{44}$). The Komlós-Szemerédi proof fits more into this survey article, for they make a general reduction first from sparse graphs to dense graphs using expander graphs (see next subsection), and then use the Regularity Lemma to handle dense graphs (as mentioned in the previous subsection). This way, they get a better constant (but the proof, as most proof using the Regularity Lemma, only works for large t): $c_1 = \sqrt{2}$, which is within a factor 2 of the truth, since a simple example of Łuczak shows that the upper bound $f(t) \leq c_2\sqrt{t}$ holds with $c_2 = 8/3$. This reduction to dense graphs will probably lead to the determination of the best c_1, too.

7.6 Expander graphs

Regular pairs are random-looking graphs in which the *number of edges* between any two (large) sets is about what it is expected to be. A much weaker notion is expansion. The bipartite graph $G = (A, B, E)$ is a weak ε-expander if, for any $X \subset A, |X| > \varepsilon|A|, \subset B, |Y| > \varepsilon|B|$, there is *at least one edge* between X and Y. This is the same to say that $X \subset A$, $|X| > \varepsilon|A|$ imply that $|N(X)| \geq (1-\varepsilon)|B|$. This notion, and especially stronger versions of expansion (in which even smaller sets X have relatively large neighbourhoods) proved to be very useful in Computer Theory. While use of expanders is certainly preferable to the use of the Regularity Lemma, expanders often don't have enough power to replace the Regularity Lemma in proofs.

An example of using expander graphs for reducing a sparse graph problem to a dense graph problem (but then the R.L is applied for the dense case) was mentioned in the previous subsection. Here we only state the self-contained theorem that is behind this reduction. Throughout this subsection $\varepsilon(.)$ denotes functions $\varepsilon : [1, \infty) \to [0, 1]$ such that $\int_1^\infty (\varepsilon(u)/u)du < \infty$, and, for given $\varepsilon(.)$, we write

$$\varepsilon_0 = \max\left\{\sup_u \varepsilon(u),\ \int_1^\infty \frac{\varepsilon(u)du}{u}\right\}.$$

Definition 7.11. *Given a function ε and a threshold x_0, a graph $G = (V, E)$ is an ε-expander if*

$$\frac{|\partial X|}{|X|} \geq \varepsilon(|X|) \tag{2}$$

for all subsets $X \subset V$, $x_0 \leq |X| \leq |V|/2$.

Theorem 7.12 (Expander subgraphs – Komlós-Szemerédi 1993 [89, 90]). *Let*

$$\varepsilon(x) \quad \text{monotone decreasing and} \quad x\varepsilon(x) \quad \text{monotone increasing for} \quad x \geq x_0, \tag{3}$$

and assume $\varepsilon_0 \leq 1/4$. Then every graph G has a subgraph $H = (V, E)$ with

$$t(H) \geq t(G)/(1 + 4\varepsilon_0) \quad \text{and} \quad \delta(H) \geq t(H)/2 \tag{4}$$

which is an ε-expander.

Example: the papers [89, 90] used $\varepsilon(x) = 1/(\log^2(x/t(G)))$.

The proof of Theorem 7.12 is fairly simple but it is using a complicated graph functional: We select a subgraph H of G that is ψ-maximal with respect to the graph-functional

$$\psi(G) = t(G)(1 + \varphi(v(G))),$$

where the function $\varphi(x)$ is defined by

$$\varphi(x) = 4\int_x^\infty \frac{\varepsilon(u)du}{u} \quad (x \geq 1).$$

Then we apply the following simple lemma.

Lemma 7.13. *Assume (3): Let $\varepsilon(x) \downarrow$ and $x\varepsilon(x) \uparrow$ for $x \geq x_0$, let*

$$\varphi(x) = c\int_x^\infty \frac{\varepsilon(u)du}{u} \quad (x \geq 1)$$

with $4 \leq c \leq 1/\varepsilon_0$, and define

$$\psi(G) = t(G)(1 + \varphi(v(G))). \tag{5}$$

Then every ψ-maximal graph is an ε-expander.

7.7 Covering transversals in multipartite graphs

Definition 7.14 (r-transversals, ε-regular cylinders). *Let $G = (V, E)$ be a k-partite graph with classes $V_1, \ldots, V_k$. A subset of $W_1 \times \ldots W_k$ of $V_1 \times \ldots V_k$ where $W_i \subset V_i$ is a* **cylinder**. *A cylinder is ε-regular if in G all the pairs (W_i, W_j) are ε-regular.*

The following theorem is from the paper of Alon, Duke, Leffman, Rödl and Yuster [3].

Theorem 7.15. *(Lemma 5.1 in [3]) For every $\varepsilon > 0$ there exists a K such that if $G(A_1, A_2, \dots, A_r)$, $|A_1| = \dots = |A_r|$, is an r-partite graph, then, for some $k \le K$, one can partition the Cartesian product $\times A_i$ as $\times A_i = \bigcup_{j<k} \times A_{i,j}$ so that all but εn^r r-transversals are covered by ε-regular pairs. Furthermore, $K < 4^{\binom{r}{2}/\varepsilon^5}$.*

As it is remarked in [3], a similar lemma was proved by Eaton and Rödl [35].

8 Strengthening the Regularity Lemma

8.1 Sparse-graph versions of the Regularity Lemma

It would be very important to find extensions of the Regularity Lemma for sparse graphs, e.g., for graphs where we assume only that

$$e(G_n) > cn^{2-\alpha},$$

for some positive constants c and α. However, we do not really know much about this. There is a new and promising development though. Y. Kohayakawa [79] and V. Rödl [106] independently proved a version of the Regularity Lemma in 1993 which can be regarded as a Regularity Lemma for sparse graphs. (Rödl's result seems to be unpublished but in [75] it is remarked that V. Rödl has also found this lemma.) As Kohayakawa puts it: "Our result deals with subgraphs of pseudo-random graphs." He (with co-authors) has also found some interesting applications of this theorem in Ramsey theory and in Anti-Ramsey theory, (see e.g. [74, 75, 76, 77, 80, 81, 82]).

To formulate the Kohayakawa–Rödl Regularity Lemma we need the following definitions.

Definition 8.1. *A graph $G = G_n$ is (P_0, η)-uniform for a partition P_0 of $V(G_n)$ if for some $p \in [0,1]$ we have*

$$|e_G(U,V) - p|U||V|| \le \eta p|U||V|,$$

whenever $|U|, |V| > \eta n$ and either P_0 is trivial, U, V are disjoint, or U, V belong to different parts of P_0.

Definition 8.2. *A partition $Q = (C_0, C_1, \dots, C_k)$ of $V(G_n)$ is (ε, k)-equitable if $|C_0| < \varepsilon n$ and $|C_1| = \dots = |C_k|$.*

Notation.

$$d_{H,G}(U,V) = \begin{cases} e_H(U,V)/e_G(U,V) & \text{if } e_G(U,V) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 8.3. *We call a pair* (U, V) (ε, H, G)*-regular if for all* $U' \subset U$ *and* $W' \subset W$ *with* $|U'| \geq \varepsilon |U|$ *and* $|W'| \geq \varepsilon |W|$*, we have*

$$|d_{H,G}(U, W) - d_{H,G}(U', W')| \leq \varepsilon.$$

Theorem 8.4 (Kohayakawa 1993 [79]). *Let* ε *and* $k_0, \ell > 1$ *be fixed. Then there are constants* $\eta > 0$ *and* $K_0 > k_0$ *with the following properties. For any* (P_0, η)*-uniform graph* $G = G_n$*, where* $P_0 = (V_i)_i^\ell$ *is a partition of* $V = V(G)$*, if* $H \subset G$ *is a spanning subgraph of* G*, then there exists an* (ε, H, G)*-regular,* (ε, k)*-equitable partition of* V *refining* P_0*, with* $k \leq k_0 \leq K_0$*.*

8.2 A hypergraph version of the Regularity Lemma

Of course, after having the powerful Szemerédi Lemma one would like to know if it can be generalized (a) to sparse graphs, (b) to hypergraphs.

One can easily formulate fake hypergraph regularity lemmas by mindlessly generalizing the original Regularity Lemma. The real question is if one can find a powerful hypergraph lemma which can be used to prove theorems which do not follow from an application of the ordinary Regularity Lemma.

The first such result was announced by Frankl and Rödl [64]. The authors write:

> "... We hope that this will prove to be nearly as useful as Szemerédi's theorem. So far we have found two applications, proof of a conjecture of Erdős concerning Turán type problems [42, 62] and giving an alternative condition for quasirandomness [25]. Proofs of these applications will be the subject of a subsequent paper."

One problem with the hypergraph version is that one feels that there must be more than one possible generalizations. When regarding 3-graphs, one can think of partitioning the vertices or the pairs of vertices. And when one has various forms, sometimes it is difficult to tell their relation.

Later we will discuss the problem of quasi-random hypergraphs. This led Fan Chung to her formulation of the hypergraph–regularity lemma stated below. Since the paper of Frankl and Rödl is somewhat concise and not too easy to read, we restrict ourselves below to formulating Fan Chung's [25] hypergraph regularity lemma.

This version has 2 parameters: k and $r < k$. The r-tuples are partitioned into t classes forming t r-uniform hypergraphs $S_1, \ldots, S_t$ and then a k-uniform hypergraph H_n is fixed and some densities $d(A_1, \ldots, A_{\binom{k}{r}})$ are defined as follows. $A_1, \ldots, A_{\binom{k}{r}}$ is a $\binom{k}{r}$-subset of $\{S_1, \ldots, S_t\}$. We count those k-tuples $E_j \in V(H_n)$ for which each r-tuple of E_j belongs to different A_i. Let their number be $e_H(A_1, \ldots, A_{\binom{k}{r}})$. The very same quantity for the complete

k-graph $K_n^{(k)}$ is denoted by $e(A_1, \ldots, A_{\binom{k}{r}})$. Let

$$d_H(A_1, \ldots, A_{\binom{k}{r}}) = \frac{e_H(A_1, \ldots, A_{\binom{k}{r}})}{e(A_1, \ldots, A_{\binom{k}{r}})}.$$

We say that $(A_1, \ldots, A_{\binom{k}{r}})$ is $(k, r) - \varepsilon$-regular if for any choice $X_i \subset A_i$ with

$$\frac{e(X_1, \ldots, X_{\binom{k}{r}})}{e_K(A_1, \ldots, A_{\binom{k}{r}})} > \varepsilon,$$

we have

$$|d_H(X_1, \ldots, X_{\binom{k}{r}}) - d_H(A_1, \ldots, A_{\binom{k}{r}})| < \varepsilon.$$

For general k, there are $k - 1$ different versions of the Regularity Lemma. Namely, for each $1 \leq r \leq k$ the following holds:

Theorem 8.5. *Suppose $1 \leq r \leq k$. For every $\varepsilon > 0$, there exists a $K(\varepsilon) > 0$ such that for every k-graph G, $\binom{V}{r}$ can be partitioned into sets $S_1, \ldots, S_t$ for some $k < K(\varepsilon)$ so that all but at most εn^k k-tuples are contained in $E(S_{i_1}, \ldots, S_{i_{\binom{k}{r}}})$ for some $i_1, \ldots, i_{\binom{k}{r}}$ where $1 \leq i_1 < i_2 < \ldots < i_{\binom{k}{r}} \leq t$ and $\{S_{i_1}, \ldots, S_{i_{\binom{k}{r}}}\}$ is (k, ε)-regular.*

Very recently J. Pach found a weakening of the hypergraph regularity lemma along the line described as (ε, δ)-regular pairs, which he needed to prove a so called Tverberg-type results in geometry, [95].

9 Algorithmic questions

The Regularity Lemma is used in two different ways in computer science. Firstly, it is used to prove the existence of some special subconfigurations in given graphs of positive edge-density. Thus by turning the lemma from an existence-theorem into an algorithm one can transform many of the earlier existence results into relatively efficient algorithms. The first step in this direction was made by Alon, Duke, Leffman, Rödl and Yuster [3].

In the second type of use, one takes advantage of the fact that the regularity lemma provides a random-like substructure of any dense graph. We know that many algorithms fail on randomlike objects. So one can use the Regularity Lemma to prove lower bounds in complexity theory, see e.g. Maass and Turán [72]. One of these randomlike objects is the expander graph, an important structure in Theoretical Computer Science.

9.1 Two applications in computer science

A. Hajnal, W. Maass and G. Turán applied the Regularity Lemma to estimate the communicational complexity of certain graph properties [72]. We quote their abstract:

"Abstract: We prove $\vartheta(n \log n)$ bounds for the deterministic 2-way communication complexity of the graph properties CONNECTIVITY, s,t-CONNECTIVITY and BIPARTITENESS. ... The bounds imply improved lower bounds for the VLSI complexity of these decision problems and sharp bounds for a generalized decision tree model which is related to the notion of evasiveness."

Another place where the Regularity Lemma is used in estimating communicational complexity is an (electronic) paper of Pudlák and Sgall [98]. In fact, they only use the (6,3)-problem, i.e., the Ruzsa-Szemerédi theorem.

9.2 An algorithmic version of the Regularity Lemma

The Regularity Lemma being so widely applicable, it is natural to ask if for a given graph G_n and given $\varepsilon > 0$ and m one can find an ε-regular partition of G in time polynomial in n. The answer due to Alon, Duke, Lefmann Rödl and Yuster [3] is surprising, at least at first: Given a graph G, we can find regular partitions in polynomially many steps, however, if we describe this partition to someone else, he cannot verify in polynomial time that our partition is really ε-regular: he has better produce his own regular partition. This is formulated below:

Theorem 9.1. *The following decision problem is co-NP complete: Given a graph G_n with a partition $V_0, V_1, \ldots, V_k$ and an $\varepsilon > 0$. Decide if this partition is ε-regular in the sense guaranteed by the Regularity Lemma.*

Let $Mat(n)$ denote the time needed for the multiplication of two $(0,1)$ matrices of size n.

Theorem 9.2 (Constructive Regularity Lemma). *For every $\varepsilon > 0$ and every positive integer $t > 0$ there exists an integer $Q = Q(\varepsilon, t)$ such that every graph with $n > Q$ vertices has an ε-regular partition into $k+1$ classes for some $k < Q$ and such a partition can be found in $O(Mat(n))$ sequential time. The algorithm can be made parallel on an EREW with polynomially many parallel processors, and it will have $O(\log n)$ parallel running time.*

9.3 Counting subgraphs

R. Duke, H. Lefmann and V. Rödl applied a version of the Szemerédi Lemma to count various subgraphs of a graph G_n relatively fast [33]. If we wish to count the subgraphs of G_n isomorphic to some given L then no really good algorithm is known. Therefore it is reasonable to try to find an approximation algorithm. The authors in [33] do not count the graphs individually, rather they fix a **list** of all the $t = 2^{\binom{k}{2}}$ labelled subgraphs $L_1, \ldots, L_t$ and this defines a vector $\sigma_k(G)$ whose i-th component is the number of "order-isomorphic" copies of L_i in the labelled G_n, where "order-isomorphic" means that the embedding $\varphi : L \to G_n$ preserves the order of the labels as well. The aim is to approximate this vector $\sigma_k(G_n)$. Clearly, if k is relatively small compared to n then it does not matter if we try to count labelled or unlabelled copies. The main result of [33] is (logarithm is of the base 2):

Theorem 9.3 ([33]). *Let c be a constant, $0 \le c \le 1$, and n an integer with $\log\log n \le \sqrt{c \log n}$. There is an algorithm which, given a labeled graph on n vertices and an ordering of its vertices and given a list of all labeled graphs on an ordered set of k vertices, $3 \le k \le \sqrt{c\log n}$, yields a $2k(k\varepsilon)^{1/2}$-approximation to $\sigma_k(G)$ in $O(2^{\binom{k}{2}} n^{2c} Mat(n))$ sequential time, where*

$$\varepsilon = \left(\frac{2^{16}k^2 \log\log n}{c \log n} \right)^{1/21}$$

and $Mat(n)$ is the time required to multiply two $n \times n$ matrices with 0,1 entries, over the integers.

10 Regularity and randomness

10.1 Extremal subgraphs of random graphs

Answering a question of P. Erdős, L. Babai, M. Simonovits and J. Spencer [8] described the Turán type extremal graphs for random graphs:

> Given an excluded graph L and a probability p, take a random graph R_n of edge-probability p (where the edges are chosen independently) and consider all its subgraphs F_n not containing L. Find the maximum of $e(F_n)$.

Below we formulate three theorems. Theorem 10.1 deals with the simplest case, namely, when $p = 1/2$ and K_3 is excluded. Theorem 10.3 generalizes Theorem 10.1 for arbitrary 3-chromatic graphs with "critical edges", (see the definition below). Theorem 10.4 describes the asymptotically extremal structure in the general case, i.e., when an arbitrary 3-chromatic L is fixed, and though $L \subset F_n$ is not excluded, the graph F_n contains only a small number of copies of L. ([8] also contains a theorem providing a more precise description of the general situation in terms of the structure L.) We will use the expression "almost surely" in the sense "with probability $1 - o(1)$ as $n \to \infty$". In this part a p-random graph means a random graph of edge-probability p where the edges are chosen independently.

Theorem 10.1. *Let $p = 1/2$. If R_n is a p-random graph and F_n is a K_3-free subgraph of R_n containing the maximum possible number of edges, and B_n is a bipartite subgraphs of R_n having maximum possible number of edges, then $e(B_n) = e(F_n)$. Moreover, F_n is almost surely bipartite.*

Definition 10.2 (Critical edges). *Given a k-chromatic graph L, an edge e is* **critical** *if $L - e$ is $k-1$-chromatic.*

Many theorems valid for complete graphs were generalized to arbitrary L having critical edges (see e.g. [114]). Theorem 10.1 also generalizes to every 3-chromatic L containing a critical edge e, and for every probability $p > 0$.

Theorem 10.3. *Let L be a fixed 3-chromatic graph with a critical edge e (i.e., $\chi(L-e) = 2$). There exists a function $f(p)$ such that if $p \in (0,1)$ is given and $R_n \in \mathbf{G}(p)$, and if B_n is a bipartite subgraph of R_n of maximum size and F_n is an L-free subgraph of maximum size, then*

$$e(B_n) \le e(F_n) \le e(B_n) + f(p)$$

almost surely, and almost surely we can delete $f(p)$ edges of F_n so that the resulting graph is already bipartite. Furthermore, there exists a $p_0 < 1/2$ such that if $p \ge p_0$, then F_n is bipartite: $e(F_n) = e(B_n)$.

Theorem 10.3 immediately implies Theorem 10.1. One could of course ask how large $f(p)$ is as $p \to 0$. We do not know the precise answer, only that Theorem 10.3 holds with $f(p) = O(p^{-3} \log p)$.

In Theorem 10.3 we are not concerned with the exact value of p_0. Our main point is that the observed phenomenon is valid not just for $p = 1/2$, but for smaller values of p as well. We do not even know if $e(F_n) - e(B_n) \to \infty$ as $p \to 0$.

If $\chi(L) = 3$ but we do not assume that L has a critical edge, then we get similar results, having slightly more complicated forms. To formulate them we should introduce the notion of the "decomposition family" of L [113]. To keep the paper short we skip these more technical details and formulate a weaker version.

Theorem 10.4. *Let L be a given 3-chromatic graph. Let $p \in (0,1)$ be fixed and let R_n be a p-random graph. If B_n is a bipartite subgraph of R_n of maximum size and F_n is an L-free subgraph of maximum size, then almost surely*

$$e(B_n) \le e(F_n) \le e(B_n) + o(n^2)$$

and we can delete $o(n^2)$ edges of F_n so that the resulting graph is already bipartite.

The above results also generalize to r-chromatic graphs L.

10.2 Random Berge-graphs

One of the deepest questions in graphs theory seems to be the Strong Perfect Graph Conjecture. This asserts that G is a perfect graph iff neither G nor its complementary graph $\overline{G}$ contains any odd cycles of length at least 5 as induced subgraphs. A surprising result of Prömel and Steger [97] asserts that statistically this conjecture is true. Let us call odd cycles on $k \ge 5$ vertices and their complementary graphs **Berge graphs**. Let $Berge(n)$ denote the class of all labelled graphs not containing Berge graphs as induced subgraphs.

Theorem 10.5 ([97]). *Almost all Berge graphs are perfect.*

Definition 10.6 (Generalized split graphs). *A graph G on the vertex set V is a Generalized Split Graph if V can be partitioned into V_1 and V_2 so that*
- either $G[V_1]$ is the union of pairwise disjoint cliques and V_2 and V_2 induced a clique in G
- or the above condition holds for the complementary graph $\overline{G}$.

Let $\mathcal{S}$ denote the family of generalized split graphs and $\mathcal{F}_5$ denote the class of graphs not containing an induced C_5. Prömel and Steger show that

(a) all the generalized split graphs are perfect, and therefore

$$\mathcal{S}(n) \subset Perf(n) \subset \mathcal{F}_5(n).$$

(b) Almost all graphs in $\mathcal{F}_5(n)$ are split graphs.

This implies

$$\mathcal{S}(n) \subset Berge(n) \subset Perf(n) \subset \mathcal{F}_5(n),$$

and that all these families have asymptotically the same cardinality. The proof uses the Regularity Lemma.

10.3 Quasi-randomness and the Regularity Lemma

Quasi-random structures have been investigated by several authors, among others, by Thomason [123], Chung, Graham, Wilson, [26]. For graphs, Simonovits and Sós [116] have shown that quasi-randomness can also be characterized by using the Regularity Lemma. Fan Chung [25] generalized their results to hypergraphs.

Let $N_G^*(L)$ and $N_G(L)$ denote the number of induced and not necessarily induced copies of L in G, respectively. Let $S(x,y) = V(G_n) - (N(x)\Delta N(y))$, the set of vertices joined to both x and y in the same way, let $N(x,y) = N(x) \cap N(y)$: the set of common neighbours of x and y. We start with the Chung-Graham-Wilson theorem in which various properties are listed all of which are almost surely true for random graphs and which are very natural properties of random graphs. The theorem asserts that even if we do not assume that a sequence (G_n) is a random graph sequence, the properties listed below are equivalent.

Theorem 10.7 (Chung-Graham-Wilson [26]). *For any graph sequence (G_n) the following properties are equivalent:*

$\mathbf{P}_1(\nu)$: *for fixed ν, for all graphs H_ν*

$$N_G^*(H_\nu) = (1+o(1))n^\nu 2^{-\binom{\nu}{2}}.$$

$\mathbf{P}_2(t)$: *Let C_t denote the cycle of length t. Let $t \geq 4$ be even.*

$$e(G_n) \geq \frac{1}{4}n^2 + o(n^2) \qquad \text{and} \qquad N_G(C_t) \leq \left(\frac{n}{2}\right)^t + o(n^t).$$

P$_3$: $\quad e(G_n) \geq \frac{1}{4}n^2 + o(n^2), \qquad \lambda_1(G_n) = \frac{1}{2}n + o(n)$ and $\lambda_2(G_n) = o(n)$, *where* $\lambda_i(G)$ *is the i-th eigenvalue of the (adjacency matrix of the) graph* G *(listed in decreasing order of modulus).*

P$_4$: *For each subset* $X \subset V$,

$$e(X) = \frac{1}{4}|X|^2 + o(n^2).$$

P$_5$: *For each subset* $X \subset V$, $\quad |X| = \lfloor \frac{n}{2} \rfloor \quad$ *we have* $\quad e(X) = \left(\frac{1}{16}n^2 + o(n^2)\right)$.

P$_6$: $\sum_{x,y \in V} \left| |S(x,y)| - \frac{n}{2} \right| = o(n^3)$.

P$_7$: $\sum_{x,y \in V} \left| |N(x,y)| - \frac{n}{4} \right| = o(n^3)$.

Obviously, $\mathbf{P}_1(\nu)$ says that the graph G_n contains each subgraph with the same frequency as a random graph. In $\mathbf{P}_2(t)$ we restrict ourselves to not necessarily induced even cycles. The difference between the role of the odd and even cycles is explained in [26]. The eigenvalue property is also very natural, knowing the connection between the structural properties of graphs and their eigenvalues. The other properties are self-explanatory.

Simonovits and Sós formulated a graph property which also proved to be a quasi-random property.

P$_S$: For every $\varepsilon > 0$ and κ there exist two integers, $k(\varepsilon, \kappa)$ and $n_0(\varepsilon, \kappa)$ such that for $n \geq n_0$, G_n has a Szemerédi-partition with parameters ε and κ and k classes $U_1, \ldots, U_k$, with $\kappa \leq k \leq k(\varepsilon, \kappa)$, so that

$$(U_i, U_j) \text{ is } \varepsilon - \text{regular, and} \qquad \left| d(U_i, U_j) - \frac{1}{2} \right| < \varepsilon$$

holds for all but at most εk^2 pairs (i,j), $1 \leq i,j \leq k$.

It is easy to see that if (G_n) is a random graph sequence of probability 1/2, then $\mathbf{P}_S$ holds for (G_n), almost surely. Simonovits and Sós [116] proved that $\mathbf{P}_S$ is a quasi-random property, i.e. $\mathbf{P}_S \iff \mathbf{P}_i$ for $1 \leq i \leq 7$. In fact, they proved some stronger results, but we skip the details.

F.R.K. Chung generalized these results to hypergraphs [25].

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